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Sous-schémas en groupes paraboliques et variétés homogènes en petites caractéristiques

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CHAPTER 1

Introduction

Au coeur de mon travail de thèse se trouve la classification des variétés algébriques projectives homogènes rationnelles sur un corps de base k algébriquement clos. L'un des objectifs est notamment d'obtenir une description qui soit la moins dépendante possible de la caractéristique de k. Ce manuscrit reprend le contenu des articles [**Mac1**] et [**Mac2**], pour en fournir une exposition plus unifiée.

Ces variétés homogènes représentent – à côté, par exemple, des variétés toriques – l'une des rares classes d'objets de la géométrie algébrique sur lesquelles on peut effectuer des calculs explicites à l'aide d'un dictionnaire combinatoire assez puissant, explicite et bien étudié (dont une référence exhaustive est [**Jan**], qui se concentre surtout sur le point de vue des représentations). Ce dictionnaire permet par exemple de tester la validité d'une conjecture, ou de donner des contre-exemples explicites pour un certain phénomène. Pour en citer quelques-uns, concernant le théorème d'annulation de Kodaira pour les fibrés en droites amples, nous mentionnons [**Lau1**, Theorem 5.2], [**Kol**, V.1.4.3], [**LR**] et [**Tot**, Theorem 2.1, Theorem 3.1].

Pour commencer, il est important de souligner qu'une classification complète et uniforme de ces variétés est connue depuis longtemps en caractéristique zéro, et avait déjà été obtenue en 1993 par Wenzel dans [**Wen**], et reprise par Haboush et Lauritzen dans [**HL**], en caractéristique au moins égale à 5. Les cas des petites caractéristiques – deux et trois – restaient ouverts. Nous verrons en effet qu'ils permettent l'existence d'objets que l'on pourrait qualifier d'exotiques.

Naturellement, la plupart des notions et des résultats qui nous intéressent proviennent du cadre de la géométrie algébrique classique sur un corps algébriquement clos k. Néanmoins, il est utile de travailler avec des schémas de type fini sur k et dans ce contexte, le mot *groupe algébrique* désigne tout schéma en groupes de type fini sur k. Ce point de vue a son importance lorsqu'on utilise des termes tels que *noyau* d'un homomorphisme de groupes, ou *intersection* de sous-schémas : ces notions doivent toutes être comprises au sens schématique. En ce qui concerne les groupes algébriques, nos références principales sont [**Bor**] et [**Spr**] pour la théorie classique sur un corps algébriquement clos. Comme référence plus récente adoptant le point de vue des schémas en groupes, nous suivons [**Mil**].

Tout d'abord, il est impératif de définir ce qu'on entend par *variété projective homogène rationnelle*. Nous supposons que tous les objets et tous les morphismes – sauf mention expresse du contraire – sont définis sur le corps de base. Nous appelons *variété* un schéma de type fini, séparé et intègre sur k et *variété projective* une variété admettant une immersion fermée dans un certain espace projectif de dimension N, noté $\mathbf{P}^N$. La condition d'homogénéité d'une variété X est définie par l'existence d'une action transitive d'un groupe algébrique lisse et connexe sur X. La condition de rationalité, c'est-à-dire de l'existence d'un ouvert dense isomorphe à un ouvert de l'espace affine, ou encore d'une application birationnelle vers l'espace projectif, est purement géométrique, et se traduit par le fait que le groupe ci-dessus est affine ; en d'autres termes, il est isomorphe à un sous-groupe fermé d'un certain GL_n.

Nous commençons dès lors par simplifier notre problème de classification, en nous restreignant aux groupes algébriques semi-simples ; cela permet d'accéder au dictionnaire combinatoire fourni par les systèmes de racines. Pour cela, quelques définitions préliminaires supplémentaires sont nécessaires.

1.1. Histoire du problème

1.1.1. Racines et paraboliques réduits. Un *tore* est un groupe algébrique T isomorphe à une puissance $\mathbf{G}_m^r$ du groupe multiplicatif. Un *sous-groupe de Borel* d'un groupe algébrique affine lisse G est un sous-groupe lisse, connexe, résoluble et maximal pour ces propriétés. Un *sous-groupe parabolique* est un sous-groupe P de G tel que le quotient G/P soit projectif. Le *radical* $R(G)$ d'un groupe algébrique G est le plus grand sous-groupe lisse connexe résoluble distingué dans G. Un groupe *semi-simple* est un groupe algébrique affine lisse connexe dont le radical est réduit à l'élément neutre. De façon analogue, le *radical unipotent* $R_u(G)$ est le plus grand sous-groupe lisse connexe unipotent distingué dans G. Un groupe est dit *réductif* s'il est affine, lisse, connexe et de radical unipotent trivial.

Un résultat de structure fondamental (voir [Sal, Théorème 5.2]) affirme que toute variété X projective homogène rationnelle peut s'écrire sous la forme

$$X = G/P,$$

où G est un groupe semi-simple et simplement connexe et P un sous-groupe parabolique de G. En effet, le résultat de de Salas permet d'abord d'écarter les variétés abeliennes, ensuite de supposer G semi-simple adjoint et donc, comme tout parabolique contient le centre, de supposer G semi-simple et simplement connexe.

De plus, les couples (B, T) - où B est un sous-groupe de Borel de G et T un tore maximal contenu dans B - sont tous $G(k)$-conjugués dans G. Enfin, les sous-groupes paraboliques de G sont précisément ceux qui contiennent un sous-groupe de Borel. La dimension du tore maximal T est appelé le *rang* du groupe G.

Cela signifie qu'on peut fixer un sous-groupe de Borel et se ramener, à conjugaison près, à l'étude des sous-groupes de G qui le contiennent. Nous fixons donc un groupe semi-simple et simplement connexe

$$G \supset B \supset T$$

muni d'un sous-groupe de Borel et d'un tore maximal.

À présent, nous définissons les objets combinatoires qui vont nous permettre de décrire les sous-groupes paraboliques. Nous considérons l'action de T sur le groupe G donnée par la conjugaison. Elle induit une action du tore T sur l'espace tangent à G en son élément neutre. Ce dernier a une structure d'algèbre de Lie : c'est un espace vectoriel sur k, muni d'une opération de crochet, qui est bilinéaire, antisymétrique et qui vérifie l'identité de Jacobi. On l'appelle algèbre de Lie de G et on le note

$$\operatorname{Lie} G.$$

Or, toute représentation de T (c'est-à-dire, toute action de T sur un espace vectoriel de dimension finie) est diagonalisable ; autrement dit, elle admet une décomposition unique en somme directe d'espaces propres. Nous pouvons donc écrire

$$\operatorname{Lie} G = \operatorname{Lie} T \oplus \left(\bigoplus_{\gamma \in \Phi} \mathfrak{g}_\gamma \right),$$

où le sous-espace des points fixes de l'action coïncide avec $\operatorname{Lie} T$. L'ensemble fini des valeurs propres

$$\Phi = \Phi(G, T)$$

est donné par les poids non triviaux du tore

$$\gamma\colon T \to \mathbf{G}_m$$

tels que l'espace de poids correspondant

$$\mathfrak{g}_\gamma := \{X \in \operatorname{Lie} G\colon\ t \cdot X = \gamma(t)X,\ \text{pour tout } t \in T\}$$

soit non nul. Ces poids sont appelés les *racines* de G par rapport au tore maximal T. Fait important, les espaces radiciels $\mathfrak{g}_\gamma$ sont de dimension 1 et correspondent à des copies du groupe additif (noté $\mathbf{G}_a$) dans G ; plus précisément, il existe des isomorphismes T-équivariants

$$u_\gamma\colon \mathbf{G}_a \xrightarrow{\sim} U_\gamma \subset G,$$

de sorte que l'action de T est donnée par

$$t \cdot u_\gamma(x) = t u_\gamma(x) t^{-1} = u_\gamma(\gamma(t)x) \text{ pour tout } t \in T,$$

où $\mathbf{G}_m$ agit sur $\mathbf{G}_a$ par multiplication. Nous notons Φ^+ le sous-ensemble des racines positives associées au sous-groupe de Borel fixé B ; par définition, une racine γ est positive si et seulement si le sous-groupe U_γ qui lui correspond est contenu dans B. De plus, une fois le sous-groupe de Borel choisi, il existe une unique *base* de racines dites *simples*, à savoir un sous-ensemble $\Delta \subset \Phi^+$ tel que toute racine positive s'écrit de façon unique comme combinaison linéaire à coefficients entiers positifs d'éléments de Δ. Le groupe G est engendré par le tore maximal, ainsi que par les sous-groupes radiciels associés aux racines simples et à leurs opposées.

Pour rendre plus compréhensible le formalisme que nous venons d'introduire, nous l'illustrons sur un exemple fondamental : le groupe $G = \mathrm{SL}_n$, avec pour sous-groupe de Borel

B celui donné par les matrices triangulaires supérieures de déterminant 1, et pour tore maximal T le tore des matrices diagonales

$$T \ni t = \operatorname{diag}(t_1, \ldots, t_n), \quad t_1 \cdots t_n = 1$$

L'algèbre de Lie de G se décompose donc de la manière suivante :

$$\operatorname{Lie} G = \operatorname{Lie} T \oplus \left(\bigoplus_{i \neq j} kE_{ij} \right),$$

où la matrice E_{ij} a un unique coefficient non nul, qui vaut 1 à la position (i, j). Notant ε_i le caractère envoyant t sur t_i, on voit que T agit sur la droite kE_{ij} avec le poids $\varepsilon_i - \varepsilon_j$, donc

$$\Phi = \{\varepsilon_i - \varepsilon_j, i \neq j\} \supset \Phi^+ = \{\varepsilon_i - \varepsilon_j, i < j\} \supset \Delta = \{\varepsilon_i - \varepsilon_{i+1}, 1 \leq i \leq n-1\}.$$

Nous sommes maintenant en mesure d'énoncer la classification des sous-groupes paraboliques réduits (c'est-à-dire, ceux dont le schéma sous-jacent est réduit). En réalité, de par leur structure additionnelle de groupes, les groupes algébriques sur k sont réduits si et seulement s'ils sont lisses. Pour $\alpha \in \Delta$, nous notons

$$P^\alpha$$

le sous-groupe parabolique et maximal parmi les paraboliques réduits contenant B et ne contenant pas $U_{-\alpha}$. Il est engendré par B et par les $U_{-\beta}$, avec β décrivant toutes les racines simples *sauf* α.

Avec les notations introduites ci-dessus, le sous-groupe (réduit, maximal) associé à la racine simple $\varepsilon_m - \varepsilon_{m+1}$ est le sous-groupe P_m suivant :

$$T = \left\{ \begin{pmatrix} * & & & & \\ & * & & & \\ & & \ddots & & \\ & & & * & \\ & & & & * \end{pmatrix} \right\} \subset B = \left\{ \begin{pmatrix} * & & \cdots & & * \\ & * & & & \\ & & \ddots & & \vdots \\ & & & * & \\ & & & & * \end{pmatrix} \right\} \subset P_m := \left\{ \begin{pmatrix} * & * & \cdots & & * \\ * & * & \cdots & & * \\ & & * & \cdots & * \\ & & \vdots & \ddots & \vdots \\ & & * & \cdots & * \end{pmatrix} \right\},$$

où les matrices dans P_m sont triangulaires par blocs, avec blocs diagonaux de taille respective m et $(n - m)$.

Pour tout groupe semi-simple G, en toute caractéristique, il existe une bijection entre les sous-groupes paraboliques *réduits* de G contenant B et les sous-ensembles de l'ensemble des racines simples Δ. Cette bijection est donnée explicitement par

$$\{\text{sous-ensembles de } \Delta\} \longrightarrow \{G \supset P \supset B\}, \quad I \longmapsto P_I := \bigcap_{\alpha \in \Delta \setminus I} P^\alpha. \tag{1.1.1}$$

Autrement dit, un sous-groupe parabolique réduit P est déterminé par la base du système de racines d'un sous-groupe de Levi. Ce que nous entendons par *sous-groupe de Levi* est un sous-groupe lisse, connexe et réductif de G (unique à conjugaison près) tel que P soit le produit semi-direct de ce sous-groupe et du radical unipotent $R_u(P)$. Il existe un unique sous-groupe de Levi contenant T ; il est engendré par T et par les sous-groupes

$$U_\alpha \quad \text{et} \quad U_{-\alpha},$$

contenus dans P. Cela nous permet de définir aussi le sous-groupe parabolique *opposé* de P, noté P^- : il est engendré par le Borel opposé (c'est-à-dire, celui associé à l'ensemble de racines $\Phi\backslash\Phi^+$) et par le sous-groupe de Levi de P.

En particulier, lorsqu'on travaille sur un corps de *caractéristique nulle*, tout groupe algébrique étant lisse, la bijection ci-dessus nous permet de classifier *tous* les sous-groupes paraboliques.

1.1.2. En caractéristique positive. Sur un corps de caractéristique $p > 0$, les sous-groupes paraboliques *peuvent ne pas être réduits*. Tout groupe algébrique H a un plus grand sous-groupe réduit, noté H_{red}. De plus, avant de s'aventurer dans le monde de la caractéristique positive, il est essentiel de rappeler que, comme il existe des groupes non lisses, un groupe algébrique n'est pas déterminé par son espace topologique sous-jacent. Si un groupe a comme espace sous-jacent un point, il est dit *infinitésimal*.

Une famille fondamentale de groupes algébriques infinitésimaux, qui jouent un rôle important dans cette thèse, est donnée par les noyaux de Frobenius. Nous notons F le morphisme de Frobenius du corps k ; pour A une k-algèbre, nous notons $A^{(1)}$ le produit tensoriel $A \otimes_{k,F} k$. Autrement dit, la structure d'algèbre de $A^{(1)}$ est donnée par

$$t \cdot a = t^p a, \quad \text{pour tout } t \in k,\, a \in A^{(1)}.$$

L'*homomorphisme de Frobenius* de A est le morphisme de k-algèbres

$$F_A \colon A^{(1)} \longrightarrow A, \quad a \otimes t \longmapsto ta^p,$$

défini par la propriété universelle du produit tensoriel. Ensuite, nous considérons le schéma affine $X = \operatorname{Spec} A$ et nous notons $X^{(1)}$ le spectre de $A^{(1)}$. Le morphisme de k-schémas associé à F_A est appelé le *morphisme de Frobenius* de X. Nous le notons

$$F_X \colon X \longrightarrow X^{(1)}.$$

En répétant une telle construction, nous définissons pour tout entier naturel m le m-ième morphisme de Frobenius *iteré*, que nous notons

$$F_X^m \colon X \longrightarrow X^{(m)}.$$

Pour un groupe algébrique affine G, le schéma $G^{(1)}$ est aussi un groupe algébrique ; de plus, le morphisme F_G^m est un morphisme de groupes algébriques pour tout m. Nous notons

$${}_mG$$

son noyau. Par définition, il s'agit d'un sous-groupe infinitésimal de G.

En raison de la lissité des groupes algébriques, en caractéristique nulle, nous pouvons récupérer beaucoup d'informations sur un sous-groupe à partir de son algèbre de Lie ; à savoir, si H est un sous-groupe de G, tous deux sont connexes et ont la même algèbre de Lie, alors ils doivent coïncider. En caractéristique positive, cela n'est clairement pas suffisant, car l'algèbre de Lie d'un groupe algébrique coïncide avec celle de son noyau de Frobenius. Pour faire face à ce problème, nous devons introduire plus de structure. Une *p-algèbre de Lie*, ou algèbre de Lie *restreinte*, est une algèbre de Lie munie d'une p-application. Cette dernière généralise à la fois la notion de Frobenius (d'une k-algèbre associative) et la notion de puissance p-ième des dérivations à valeurs dans k.

Un résultat crucial, nous aidant à voir clairement combien d'informations nous pouvons récupérer à partir de l'algèbre de Lie, est le suivant. Pour tout groupe algébrique G, il y a une équivalence de catégories entre les sous-groupes de G *tués par le Frobenius* et les *p-sous-algèbres de Lie* de $\operatorname{Lie} G$. Cette équivalence est fréquemment utilisée tout au long de cette thèse.

En revenant aux paraboliques, nous commençons par présenter un premier exemple, donné par une famille de variétés relativement simples, définies comme des versions *tordues* de la variété d'incidence dans le plan projectif. Plus précisément, nous considérons

$$\mathrm{SL}_3 \circlearrowright X_m := \{x_0^{p^m} y_0 + x_1^{p^m} y_1 + x_2^{p^m} y_2 = 0\} \subset \mathbf{P}^2 \times \mathbf{P}^2, \tag{1.1.2}$$

où m est un entier positif. De plus, nous nous intéressons à l'action de $G = \mathrm{SL}_3$ sur $\mathbf{P}^2 \times \mathbf{P}^2$, agissant comme suit :

$$A \cdot (x, y) = (Ax,\, {}^t B^{-1} y), \quad \text{où } b_{ij} := a_{ij}^p.$$

De cette manière, la variété X_m est préservée par l'action, et on vérifie que c'est une variété SL_3-homogène. Si on prend comme point de base celui de cordonnées

$$([1:0:0], [0:0:1]) \in X_m,$$

un calcul explicite de son stabilisateur $P_{(m)}$ donne

$$P_{(m)} = \left\{ \begin{pmatrix} a & b & c \\ 0 & e & f \\ 0 & h & i \end{pmatrix} \in \mathrm{SL}_3 \colon \; h^{p^m} = 0 \right\} \subset \mathrm{SL}_3 \,. \tag{1.1.3}$$

Remarquons que pour $m = 0$, le sous-groupe parabolique $P_{(0)}$ coïncide avec le sous-groupe de Borel B donné par les matrices triangulaires supérieures de déterminant 1. En revanche, pour tout $m \geq 1$, le sous-groupe $P_{(m)}$ est un parabolique *non réduit*.

Nous pouvons maintenant fournir un résumé des principaux résultats de structure pour les sous-groupes paraboliques, en caractéristique positive (y compris 2 et 3).

Soit P un sous-groupe parabolique d'un groupe semi-simple, simplement connexe G, de partie réduite P_{red}. Suivant [Wen], nous notons

$$U_P^- := P \cap R_u(P_{\mathrm{red}}^-)$$

son intersection avec le radical unipotent du parabolique opposé de P_{red}. Le sous-groupe U_P^- est unipotent et infinitésimal par construction. De plus, il satisfait

$$U_P^- = \prod_{\gamma \in \Phi^+ \setminus \Phi_I} (U_P^- \cap U_{-\gamma}) \quad \text{et} \quad P = U_P^- \times P_{\mathrm{red}},$$

où les deux égalités sont des isomorphismes de schémas donnés par la multiplication de G. Cela implique que P peut s'obtenir à partir de sa partie réduite P_{red}, et de ses intersections avec tous les sous-groupes radiciels contenus dans le radical unipotent opposé $R_u(P_{\mathrm{red}}^-)$. Reformulons cet énoncé de manière plus combinatoire, en introduisant une fonction numérique. Nous désignons par $\boldsymbol{\alpha}_{p^n}$ le sous-groupe du groupe additif donné par

$$\boldsymbol{\alpha}_{p^n} := \{x \in \mathbf{G}_a,\, x^{p^n} = 0\},$$

tandis que $\boldsymbol{\alpha}_{p^\infty}$ désigne $\mathbf{G}_a$.

Définition 1.1.1. Soit P un sous-groupe parabolique d'un groupe semi-simple G. La *fonction associée*

$$\varphi \colon \Phi \longrightarrow \mathbf{N} \cup \{\infty\}$$

est donnée par

$$P \cap U_{-\gamma} = u_{-\gamma}(\boldsymbol{\alpha}_{p^{\varphi(\gamma)}}), \quad \gamma \in \Phi^+.$$

En d'autres termes, toute racine positive γ (ne faisant pas partie du système de racines du sous-groupe de Levi) est envoyée sur l'entier naturel correspondant à la hauteur de $P \cap U_{-\gamma}$, où la *hauteur* d'un sous-groupe est le plus petit entier positif m tel que le morphisme de Frobenius itéré m fois l'annule. De leur côté, toutes les autres racines sont envoyées sur l'infini.

Le résultat de structure fondamental - [**Wen**, Théorème 10] - est le suivant : le sous-groupe parabolique P est uniquement déterminé par la fonction φ, sans aucune hypothèse sur la caractéristique ou sur le groupe.

1.1.3. Caractéristique au moins 5. Sous l'hypothèse additionnelle que $p \geq 5$, Wenzel [**Wen**], Haboush et Lauritzen [**HL**] montrent que tous les sous-groupes paraboliques de G peuvent être obtenus à partir de paraboliques réduits maximaux, en les épaississant avec les noyaux de Frobenius puis en prenant leurs intersections. Leur résultat est vrai aussi sous l'hypothèse que G est *simplement lacé*, c'est-à-dire qu'il ne contient aucun facteur simple isomorphe à Spin_{2n+1} - le revêtement universel du groupe spécial orthogonal SO_{2n+1} - ni isomorphe au groupe symplectique Sp_{2n}, ni isomorphe à un groupe exceptionnel de type F_4 ou de type G_2.

Plus précisément, sous les hypothèses que nous venons de mentionner, tout sous-groupe parabolique de G est de la forme

$$(1.1.4) \qquad {}_{m_1}GP^{\beta_1} \cap \ldots \cap {}_{m_r}GP^{\beta_r},$$

où $\beta_1, \ldots, \beta_r$ sont des racines simples de G et $m_1, \ldots, m_r$ des entiers positifs. Dans la suite, un parabolique de cette forme sera appelé parabolique *de type standard*.

Par exemple, le sous-groupe parabolique $P_{(m)}$ introduit dans (1.1.3) est de type standard, car il s'exprime comme

$$P_{(m)} = P^{\varepsilon_1-\varepsilon_2} \cap {}_1GP^{\varepsilon_2-\varepsilon_3}.$$

En termes de fonctions numériques, la fonction associée au sous-groupe parabolique

$${}_mGP^{\alpha}$$

envoie toutes les racines positives à l'infini, sauf celles contenant α dans leur support, qui prennent la valeur m.

La démonstration de [**Wen**] repose fortement sur les constantes de structure relatives à une base de Chevalley de l'algèbre de Lie d'un groupe semi-simple simplement connexe. Une base de Chevalley de $\operatorname{Lie} G$ est donnée par

$$\{X_\gamma \colon \gamma \in \Phi,\, H_\alpha \colon \alpha \in \Delta\},$$

où les H_α forment une base de $\operatorname{Lie} T$ (l'algèbre de Lie du tore maximal). De plus, cette base satisfait

$$\mathfrak{g}_\gamma = \operatorname{Lie} U_\gamma = kX_\gamma \quad \text{et} \quad X_\gamma = \frac{d}{du}\bigg|_{u=1}.$$

Considérons maintenant deux racines γ et δ telles que le poids $\gamma+\delta$ soit encore une racine. Soit r l'entier positif tel que $\gamma - r\delta$ soit une racine, mais $\gamma - (r+1)\delta$ ne le soit pas. La relation fondamentale qui nous intéresse est la suivante :

$$[X_\gamma, X_\delta] = \pm(r+1)X_{\gamma+\delta}.$$

L'entier $r+1$ - défini à un signe près - est appelé la *constante de structure* associée au couple (γ, δ). Par construction, de telles constantes sont indépendantes de la caractéristique et sont des entiers dont la valeur absolue est strictement inférieure à 5.

Les hypothèses faites sur la caractéristique ($p \geq 5$) ou sur le groupe (simplement lacé) garantissent toutes deux que ces constantes ne s'annulent pas sur k.

Depuis les travaux que nous venons de mentionner, la question suivante est restée ouverte pendant presque trois décennies : que se passe-t-il dans le cas d'un corps de caractéristique 2 ou 3, sous l'hypothèse que le groupe G ait au moins un facteur non simplement lacé ? Dans [**Wen**], on trouve seulement une allusion à des paraboliques qui ne sont pas de type standard ; les deux premiers exemples explicites (obtenus avec des méthodes de théories de représentations) se trouvent dans [**Lau2**, Section 3.3]. Depuis, la littérature semble avoir ignoré le problème.

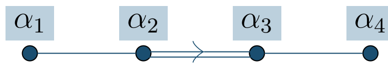
1.2. Isogénies sans facteur central

Nous pouvons restreindre notre attention à un groupe G *simple* et *simplement connexe*, ce que nous faisons dans cette section. Cette digression est indépendante et motivée par l'idée qu'une certaine classe d'isogénies va nous permettre de généraliser le rôle joué par le morphisme de Frobenius dans [**Wen**]. Une *isogénie* est un homomorphisme de groupes algébriques qui est fini et fidèlement plat. On parle d'isogénie *centrale* quand le noyau est un sous-groupe central.

1.2.1. Isogénie très spéciale. Supposons que le diagramme de Dynkin de G ait une arête de multiplicité p, c'est-à-dire que G soit de type B_n, C_n ou F_4 en caractéristique deux, ou de type G_2 en caractéristique trois. Plus précisément, pour un groupe G de rang n, on note

$$\alpha_1, \ldots, \alpha_n$$

la base Δ des racines simples, et les différents cas sont les suivants. En type B_n (c'est-à-dire quand on travaille avec le revêtement universel d'un groupe spécial orthogonal agissant sur un espace vectoriel de dimension impaire) on note α_n la racine simple *courte*. De façon duale, en type C_n (c'est-à-dire dans le cas d'un groupe symplectique) on note α_n la racine simple *longue*. Quand on travaille avec le groupe exceptionnel de type F_4, la convention que nous adoptons pour son diagramme de Dynkin est la suivante :



Finalement, pour le groupe exceptionnel de type G_2 le diagramme de Dynkin est representé comme suit :

Sous ces hypothèses, un ingrédient important est l'*isogénie très spéciale* d'un groupe simple et simplement connexe G qui est le quotient

$$\pi_G \colon G \longrightarrow \overline{G}$$

par le sous-groupe N_G de G, non central, de Frobenius trivial et minimal pour ces propriétés. Un tel sous-groupe est uniquement déterminé par son algèbre de Lie qui est définie en termes de racines simples *courtes*. Il se trouve que lorsque le diagramme de Dynkin de G a une arête de multiplicité égale à la caractéristique, un tel sous-groupe est strictement contenu dans le noyau de Frobenius. En particulier, π_G agit comme un morphisme de Frobenius sur les copies du groupe additif associées aux racines courtes, et c'est un isomorphisme sur les copies du groupe additif associées aux racines longues.

Le groupe quotient $\overline{G}$ est simple, simplement connexe et possède un système de racines dual de celui de G ; de plus, la composition

$$\pi_{\overline{G}} \circ \pi_G$$

est égale au morphisme de Frobenius de G (voir les travaux originaux de Borel et Tits [**BT**] sur le sujet, ainsi que le livre [**CGP**, Chapitre 7] pour plus de détails). Pour rendre cet objet plus concret, on mentionne ici un exemple naturel provenant de l'algèbre linéaire, qui peut être réinterprété comme une isogénie très spéciale, et qui est déjà décrit dans [**Ste**, 4.11].

On suppose que la caractéristique est égale à deux et on considère les groupes $G = \mathrm{SO}_{2n+1}$

et $G' = \mathrm{Sp}_{2n}$, relatifs respectivement aux formes quadratiques et symplectiques

$$x_0^2 + \sum_{i=1}^{n} x_i x_{n+i} \quad \text{et} \quad \sum_{i=1}^{n} \left(y_i y'_{n+i} - y_{n+i} y'_i \right).$$

La caractéristique étant égale à deux, le groupe G fixe le premier vecteur e_0 de la base canonique de k^{2n+1}, agissant ainsi sur le quotient

$$k^{2n+1}/ke_0 = k^{2n}.$$

Cela induit une isogénie

$$\psi \colon G \longrightarrow G',$$

dont le noyau est unipotent et infinitésimal. Le relèvement de ψ au revêtement universel de G donne une construction de l'isogénie très spéciale dans cet exemple.

1.2.2. Factorisation des isogénies. Le premier pas vers la généralisation de la classification aux petites caractéristiques est une propriété de factorisation, que l'on peut formuler comme suit.

Proposition 1.2.1. *(Proposition 2.5.12)*
Soit G un groupe algébrique simple et simplement connexe sur un corps algébriquement clos k de caractéristique $p > 0$. Soit $f : G \to G'$ une isogénie.
Alors il existe une factorisation unique de f comme

$$f \colon G \xrightarrow{\quad \pi \quad} \overline{G} \xrightarrow{\quad F^m_{\overline{G}} \quad} (\overline{G})^{(m)} \xrightarrow{\quad \rho \quad} G',$$

où m est un entier naturel, ρ est une isogénie centrale et π est soit l'identité, soit - lorsque le diagramme de Dynkin de G a une arête de multiplicité p - l'isogénie très spéciale π_G.

Cet énoncé nous permet de parler d'isogénies *sans facteur central.* Ces dernières sont de la forme

$$F^m \quad \text{ou} \quad F^m \circ \pi$$

où m est un entier positif. Les noyaux de cette classe d'isogénies sont les sous-groupes infinitésimaux de G qui revêtent une importance cruciale pour la classification des sous-groupes paraboliques. L'exclusion des isogénies centrales s'avère essentielle, car tout sous-groupe central de G est contenu dans le tore maximal et, par conséquent, dans tout sous-groupe parabolique. Il est fondamental de remarquer qu'il existe un ordre total sur ces noyaux par inclusion :

$$1 \subsetneq N \subsetneq {}_1G \subsetneq \ker(F \circ \pi_G) \subsetneq \dots \subsetneq {}_mG \subsetneq \ker(F^m \circ \pi_G) \subsetneq {}_{m+1}G \subsetneq \dots \tag{1.2.1}$$

Définition 1.2.2. On dit qu'un sous-groupe parabolique P est *de type quasi-standard* si on peut l'obtenir à partir de paraboliques réduits en les épaississant avec des sous-groupes distingués infinitésimaux et en les intersectant, c'est-à-dire si P est de la forme

$$\bigcap_{\alpha} (\ker \xi_\alpha) P^\alpha,$$

où chaque ξ_α est une isogénie sans facteur central.

Cette définition est la généralisation souhaitée de la définition de parabolique de type standard. Cela nous permet de réinterpréter les premiers exemples appelés *exceptionnels* dans [**Lau2**] d'une façon assez claire et concise. On se place en caractéristique $p = 2$. Le premier exemple donné dans cet article est le suivant : dans un groupe G simple, simplement connexe, de type B_2, il considère le parabolique

$$N_G P^{\alpha_1},$$

où α_1 est la racine simple courte. Le deuxième exemple est comme suit : dans un groupe G simple, simplement connexe, de type C_4, avec racine simple longue α_4, il considère le sous-groupe parabolique

$$N_G P^{\alpha_4}.$$

Il s'agit de deux sous-groupes paraboliques de type quasi-standard, mais qui ne sont pas de type standard.

1.3. Sous-groupes paraboliques avec partie réduite maximale

Nous parvenons à terminer la classification de la classe la plus simple à définir - du point de vue combinatoire - de sous-groupes paraboliques, c'est-à-dire ceux ayant une partie réduite maximale égale à P^α, où α est une racine simple de G par rapport au sous-groupe de Borel B. Dans cette partie, on peut encore une fois se ramener au cas d'un groupe G simple.

1.3.1. Cas quasi-standard. Notre résultat principal est valable pour tout type de groupe G et en toute caractéristique $p > 0$, sauf dans le cas où $p = 2$ et le groupe G est de type G_2. Pour ce dernier cas, deux nouvelles familles exotiques de sous-groupes paraboliques entrent en jeu.

Théorème A (Théorème 3.3.2). *Soit P un sous-groupe parabolique non réduit. Si P_{red} est maximal, il existe une unique isogénie $\xi \colon G \to G'$ sans facteur central tel que*

$$P = (\ker \xi) P_{red},$$

sauf si G est de type G_2, $p = 2$ et si $P_{red} = P^{\alpha_1}$, où α_1 est la racine simple courte.

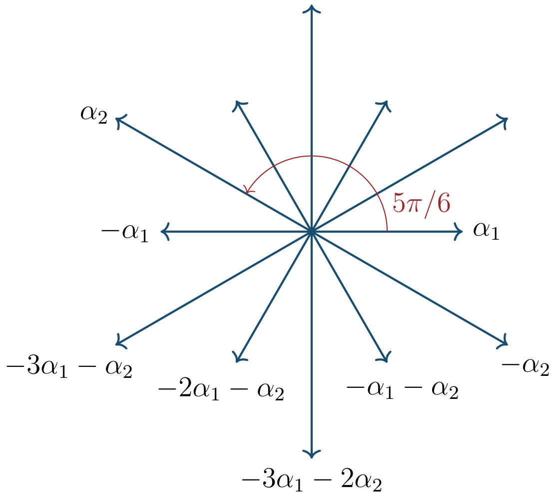
Afin de compléter cet énoncé et d'obtenir une classification uniforme, nous reviendrons rapidement sur le cas d'un groupe de type G_2 (Théorème B).

La preuve de ce résultat commence par les cas quasi-standard, pour lesquelles la stratégie de preuve est uniforme. Comme le comportement exotique en type G_2 le confirme, il n'y a pas moyen d'avoir une preuve qui utilise des arguments géométriques généraux : nous procédons donc par une analyse au cas par cas. La preuve s'articule essentiellement en trois étapes. La première consiste en quelques réductions élémentaires impliquant qu'il suffit de montrer l'assertion suivante : si la partie réduite de P est maximale et si G agit fidèlement sur G/P, alors P doit lui-même être réduit. La deuxième étape exploite la description explicite du quotient

$$\operatorname{Lie} G / \operatorname{Lie} P_{\mathrm{red}},$$

vu comme représentation d'un sous-groupe de Levi de P_{red}. Enfin, la dernière étape consiste à considérer certaines constantes de structure (choisies de manière à ce qu'elles ne s'annulent pas en fonction de la caractéristique du corps de base k) et à conclure en utilisant la notion d'isogénie très spéciale.

1.3.2. Type G_2. Nous nous concentrons ensuite sur le cas d'un corps de caractéristique 2 et d'un groupe de type G_2. Nous notons α_1 sa racine simple courte et α_2 la racine simple longue, de sorte que le système de racines soit le suivant.



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Étonnamment peut-être, la stratégie de preuve analogue à celle évoquée dans le paragraphe précédent fonctionne lorsque la partie réduite est P^{α_2}, mais échoue lorsqu'on considère l'hypothèse

$$P_{\text{red}} = P^{\alpha_1},$$

en raison de l'annulation de certaines constantes de structure. Pour expliquer ce problème, on démontre que $\operatorname{Lie} P^{\alpha_2}$ est une sous-algèbre de Lie maximale, mais qu'il existe exactement deux sous-algèbres de Lie maximales $\mathfrak{h}$ et $\mathfrak{l}$ de $\operatorname{Lie} G$ contenant strictement $\operatorname{Lie} P^{\alpha_1}$.

Nous les décrivons explicitement et nous considérons les sous-groupes correspondants de hauteur 1 dans G qui donnent naissance à deux nouveaux sous-groupes paraboliques notés $P_{\mathfrak{h}}$ et $P_{\mathfrak{l}}$. Ces derniers sont assez différents des autres car l'algèbre de Lie de G est simple (en tant que p-algèbre de Lie). On ne peut pas les décrire comme épaississements de P^{α_1} par le noyau d'une isogénie. Ensuite, nous étudions les deux espaces homogènes correspondants, grâce à la description de G_2 en tant que groupe d'automorphismes d'une algèbre d'octonions ; voir [**SV**]. Il s'avère que $G/P_{\mathfrak{h}}$ est isomorphe à l'espace projectif $\mathbf{P}^5$, tandis que nous réalisons $G/P_{\mathfrak{l}}$ comme une section hyperplane

$$G_2 \circlearrowright \mathcal{X} \hookrightarrow Y \circlearrowleft \mathrm{Sp}_6,$$

de la variété Sp_6-homogène Y, qui paramètre les sous-espaces isotropes de dimension 3 dans un espace vectoriel symplectique de dimension 6. La construction de $\mathcal{X}$ est obtenue explicitement ; cela permet d'interpréter la variété $\mathcal{X}$ dans un cadre de théorie des représentations.

Plus précisément, nous obtenons la classification de *tous* les sous-groupes paraboliques ayant partie réduite maximale avec l'énoncé suivant.

Théorème B (Proposition 3.2.26). *Soit G de type G_2 en caractéristique deux. Alors tout sous-groupe parabolique non réduit de G ayant P^{α_1} comme partie réduite est soit de type standard, soit obtenu à partir de $P_{\mathfrak{l}}$ ou de $P_{\mathfrak{h}}$ par tiré en arrière avec un homomorphisme de Frobenius itéré.*

1.3.3. Interprétation géométrique. Il est naturel à ce stade de revenir à notre motivation initiale et de chercher à interpréter cette description d'un point de vue géométrique. Il s'avère que les sous-groupes paraboliques dont la partie réduite est maximale correspondent aux variétés homogènes dont le groupe de Picard est isomorphe à $\mathbf{Z}$. Nous reviendrons dans la suite sur une description plus détaillée ; pour l'instant, il suffit de savoir que le groupe de Picard de G/P est abélien, libre et de rang égal au nombre de racines simples qui ne sont *pas* dans le sous-groupe de Levi de la partie réduite de P. Cela justifie aussi le fait que l'on exprime (voir (1.1.1) ci-dessus) chaque parabolique réduit en termes des racines simples que sa partie réduite ne contient pas (ce qui s'éloigne des notations standard dans le domaine).

Théorème C (Version géométrique : Théorème 3.1.1 et Proposition 3.2.19). *Soit X une variété homogène projective, en caractéristique quelconque, dont le groupe de Picard est isomorphe à $\mathbf{Z}$. Alors X est soit isomorphe à une variété homogène de stabilisateur un parabolique* réduit *maximal, soit isomorphe à la variété G_2-homogène $\mathcal{X}$; ce deuxième cas ne se produit qu'en caractéristique 2.*

1.4. Classification complète

Finalement, mon travail a permis d'aller plus loin et d'obtenir la liste complète des sous-groupes paraboliques, pour tout type et toute caractéristique. On travaille ici avec un groupe G semi-simple et simplement connexe.

Théorème D (Théorème 4.1.1). *Soit P un sous-groupe parabolique de G et soient $\beta_1, \ldots, \beta_r$ les racines simples de G telles que*

$$P_{red} = \bigcap_{i=1}^{r} P^{\beta_i}.$$

Alors

$$P = \bigcap_{i=1}^{r} Q^i,$$

où Q^i est le plus petit sous-groupe de G contenant P et P^{β_i}.
En particulier, P est l'intersection de sous-groupes paraboliques ayant une partie réduite maximale. Ainsi, tous les paraboliques sont de type quasi-standard*, sauf dans le cas d'un corps de base de caractéristique 2 et lorsque G a un facteur de type G_2.*

Un ingrédient crucial dans la démonstration de ce résultat est le fait que les noyaux des isogénies sans facteur central sont strictement contenus les uns dans les autres, et donc ordonnés totalement par inclusion, comme déjà remarqué dans (1.2.1). Encore une fois, la preuve est différente dans le cas d'un groupe de type G_2 en caractéristique 2 et doit être traitée séparément.

Tout d'abord, nous nous ramenons à traiter le cas d'un groupe simple et simplement connexe. Nous remarquons ensuite quelques faits sur les fonctions numériques associées, définies selon la Définition 1.1.1 ci-dessus. Nous montrons ensuite le lemme suivant qui est le point clé de la preuve.

Lemme 1.4.1. *Avec les notations du Théorème D, considérons une racine simple β_i. Alors*

$$U_{-\beta_i} \cap P = U_{-\beta_i} \cap Q^i.$$

Pour démontrer ce lemme, nous faisons appel à la notion de diviseur de Schubert et de courbes de Schubert dans une variété projective homogène $X = G/P$ qui permettent de décrire de manière combinatoire le groupe de Picard et le cône des 1-cycles effectifs dans X. Cette description est donnée en détail plus tard : ce n'est pas une conséquence de la classification, mais plutôt un outil pour la démontrer. Cependant, nous choisissons de reporter cette partie dans la suite du texte afin que la preuve de la classification soit autant que possible concentrée sur la combinatoire des systèmes de racines et moins sur la géométrie. Le point important est que les sous-groupes Q^i dans l'énoncé peuvent être construits d'une manière purement géométrique comme stabilisateurs des points de certains morphismes qui contractent toutes les courbes de Schubert sauf une.

Une fois le lemme ci-dessus démontré, nous poursuivons par une analyse au cas par cas selon le diagramme de Dynkin de G et la caractéristique du corps de base.

1.5. Géométrie des G/P

L'existence de sous-groupes paraboliques non réduits en caractéristique positive autorise des comportements différents et plus riches des espaces homogènes qui leur correspondent.

Pour illustrer certains aspects géométriques, nous pouvons revenir à l'exemple (1.1.2). Pour $m \geq 1$, la variété X_m n'est pas localement rigide, c'est-à-dire que son fibré tangent admet de la cohomologie en degré 1. Cela ne peut pas arriver dans le cas d'un stabilisateur réduit : voir [**Dem**, Théorème 2]. De plus, chaque X_m étant une hypersurface, nous pouvons calculer son fibré canonique grâce à la formule d'adjonction. Nous en déduisons qu'une telle variété n'est pas de Fano lorsque $p^m > 3$; cela représente encore une fois un type de comportement qui s'éloigne de celui en caractéristique nulle. On peut mentionner le travail récent dans cette direction par [**Tot**, Theorem 2.1, Theorem 3.1] : on y trouve une famille d'exemples, remarquables car relativement simples à définir, de variétés G/P qui sont de Fano et qui ne satisfont pas le théorème d'annulation de Kodaira pour les fibrés amples (par variété de Fano, nous entendons une variété projective lisse dont le fibré anti-canonique est ample).

1.5.1. Diviseurs, courbes et contractions. Tout d'abord, nous obtenons une description explicite des classes de courbes et de diviseurs. Rappelons que les variétés sur lesquelles nous portons notre attention sont lisses, projectives et munies d'une action transitive de G. Sous ces hypothèses, il existe une stratification, connue sous le nom de décomposition de Białynicki-Birula (ce qui, dans le cas d'un stabilisateur réduit, revient à la décomposition de Bruhat, qui est mieux connue). Le travail original sur le sujet est présenté dans [**BB**] ; pour une formulation du point de vue de la théorie des schémas, voir [**Mil**, Theorem 13.47]. Cette décomposition consiste à écrire la variété $X = G/P$ comme une réunion disjointe de cellules, isomorphes à des espaces affines, indexées par l'ensemble fini des points fixes de l'action de T sur X, où T est le tore maximal que l'on se donne dans G.

En particulier, il existe une unique cellule ouverte

$$X^{-} \subset X$$

isomorphe à l'espace affine de dimension égale à la dimension de X. Cette cellule est obtenue comme B^{-}-orbite du point de base de X, où B^{-} est le sous-groupe de Borel opposé à B (c'est-à-dire, associé aux racines négatives). Les composantes irréductibles de $X \backslash X^{-}$ sont indexées par les racines simples α qui ne sont pas dans le sous-groupe de Levi de P_{red}. En prenant leur adhérence de Zariski, nous obtenons une famille finie de diviseurs D_α effectifs, donc globalement engendrés, dont les classes d'équivalence linéaire forment une base du groupe de Picard de X. On les appelle les *diviseurs de Schubert* de X. Nous construisons de façon analogue les objets duaux, c'est-à-dire les *courbes* de Schubert C_β, qui sont des courbes rationnelles lisses B-stables, indexées par le même ensemble de racines simples, et qui donnent une base du cône des 1-cycles effectifs dans X. De plus, nous vérifions que les nombres d'intersection correspondants satisfont

$$D_\alpha \cdot C_\beta = \delta_{\alpha\beta}.$$

À partir de chaque diviseur D_α, nous définissons un morphisme

$$f_\alpha \colon X \longrightarrow G/Q^\alpha =: Y_\alpha. \tag{1.5.1}$$

L'application f_α est une *contraction*, c'est-à-dire qu'elle satisfait l'égalité

$$(f_\alpha)_* \mathcal{O}_X = \mathcal{O}_{Y_\alpha}.$$

De plus, f_α est l'unique contraction de source X telle que toutes les courbes de Schubert sauf C_α soient contractées en un point.

Après cette définition, donnée en termes purement géométriques, nous démontrons que Q^α est en effet le sous-groupe engendré par P et P^α, ce qui fait des Q^α les sous-groupes à partir desquels on peut retrouver P par intersection (voir Théorème D).

1.5.2. Nouveaux exemples en rang 2. Maintenant que notre attention se porte sur la géométrie des variétés homogènes, une question centrale se pose : pour un parabolique de type quasi-standard P, la variété G/P est-elle toujours isomorphe à une variété G'/P', où P' est de type standard ? Cette assertion est vérifiée pour une variété dont le groupe de Picard est isomorphe à $\mathbf{Z}$, comme conséquence du Théorème A.

Nous construisons un premier contre-exemple en rang 2. Autrement dit, en utilisant le noyau de l'isogénie très spéciale, nous définissons une famille de variétés homogènes de rang de Picard 2, dont les variétés sous-jacentes ne sont *pas de type standard*. Par cette terminologie, nous entendons qu'elles ne sont isomorphes (en tant que variétés) à aucun quotient G/P ayant pour stabilisateur un sous-groupe parabolique P de type standard. L'énoncé suivant fournit ce contre-exemple, où les conventions concernant les systèmes de racines sont toujours celles de [**Bou**].

Proposition 1.5.1 (Proposition 5.3.1)**.** *Soit G un groupe simple et simplement connexe sur un corps algébriquement clos de caractéristique 2. Soient α et β des racines simples distinctes telles que : ou bien G est de type B_n ou C_n et la paire (α, β) est de la forme (α_j, α_i) avec $i < j < n$ ou $j = n$ et $i < n-1$, ou bien G est de type F_4 et la paire (α, β) est l'une parmi*

$$(\alpha_1, \alpha_4), \quad (\alpha_2, \alpha_1), \quad (\alpha_2, \alpha_4), \quad (\alpha_3, \alpha_1), \quad (\alpha_3, \alpha_4), \quad (\alpha_4, \alpha_1).$$

Alors l'espace homogène

$$X = G/(NP^\alpha \cap P^\beta)$$

n'est pas *de type standard, où $N = N_G$ est le noyau de l'isogénie très spéciale de G.*

L'enoncé ci-dessus est assez technique, mais on peut mieux le comprendre de la façon suivante. L'hypothèse sur la caractéristique vise à assurer l'existence de l'isogénie très spéciale. L'hypothèse combinatoire sur les racines α et β est elle aussi cruciale pour éviter l'égalité

$$NP^\alpha \cap P^\beta = {}_1GP^\alpha \cap P^\beta,$$

autrement, le terme de gauche serait déjà de type standard.

La preuve de la Proposition 1.5.1 repose sur la description des courbes et des diviseurs de Schubert, ainsi que sur les résultats concernant les groupes d'automorphismes des variétés de drapeaux, établis par Demazure dans [**Dem**].

1.5.3. Conséquences de la classification. Comme première application du résultat principal (Théorème D), nous déterminons de manière explicite les sous-groupes Q^i qui apparaissent dans l'énoncé de la classification, sous l'hypothèse que le parabolique P de départ soit de type standard. Il s'avère que la description de P comme étant l'intersection des Q^i n'est pas aussi simple qu'elle pourrait sembler à première vue ; à vrai dire, même dans le cas où P est un parabolique de type standard, certains des sous-groupes Q^i peuvent (selon la combinatoire du système de racines considéré) être de type quasi-standard, voire des sous-groupes exotiques dans le cas de type G_2.

Nous considérons ensuite la question de l'existence d'au moins une contraction lisse issue d'une variété homogène $X = G/P$. Après réduction au cas où P ne contient le noyau d'aucune isogénie sans facteur central, cela revient à se demander si le stabilisateur P peut être contenu dans un sous-groupe parabolique lisse. Une telle contraction existe pour tout parabolique de type quasi-standard, mais elle peut ne pas exister dans certains cas exotiques en type G_2 et caractéristique 2.

En itérant une telle construction, nous définissons à partir de X (sous l'hypothèse que le stabilisateur P soit de type quasi-standard) une suite finie de fibrations, localement triviales pour la topologie de Zariski, dont les fibres sont de type standard, avec groupe de Picard isomorphe à $\mathbf{Z}$.

Une autre question naturelle à se poser concerne le comportement des fibrés en droites et leur positivité sur cette classe de variétés. Nous obtenons des premiers éléments de réponse : si on s'intéresse aux variétés sous-jacentes, le Théorème D affirme qu'il existe une immersion fermée

$$G/P = X \hookrightarrow \prod_i G/Q^i = Y_i \ .$$

De leur côté, chaque Y_i a par définition un groupe de Picard isomorphe à $\mathbf{Z}$. Cela signifie, grâce au Théorème A, que soit Y_i est isomorphe à une variété homogène ayant pour stabilisateur un sous-groupe parabolique réduit maximal, soit elle est isomorphe à l'unique variété G_2-homogène exotique $\mathcal{X}$. Comme première application de l'existence de cette immersion fermée, nous déduisons que tout fibré en droites ample sur X est en fait *très ample.*

1.5.4. Un résultat de finitude. Nous commençons par mentionner quelques résultats, vrais dans le cas d'un parabolique réduit, mais faux en caractéristique positive. Premièrement, si on se donne un entier positif n, il n'existe qu'un nombre fini de variétés G/P de dimension n et ayant stabilisateur réduit. De plus, toute variété de cette forme est de Fano (voir [**Kol**, V, Theorem 1.4]). Si nous autorisons le stabilisateur à être non réduit, un contre-exemple à ces deux propriétés est donné par la famille des variétés X_m définie dans (1.1.2).

La troisième et dernière propriété nécessite une définition et quelques explications supplémentaires. Une variété X, sur un corps de caractéristique $p > 0$, est dite *scindée (par le Frobenius)* si le morphisme

$$\mathcal{O}_{X^{(1)}} \longrightarrow (F_X)_* \mathcal{O}_X$$

se scinde en tant que morphisme de $\mathcal{O}_{X^{(1)}}$-modules, où

$$F_X \colon X \longrightarrow X^{(1)}$$

est le morphisme de Frobenius relatif de X. Cette propriété fournit des informations importantes sur la géométrie de X. Par exemple, si une variété X est scindée, on peut en déduire des propriétés d'annulation pour la cohomologie des fibrés en droites amples sur X ; voir [**BK**, Theorem 1.2.9]. En particulier, les variétés G/P avec P réduit sont scindées par le Frobenius. Encore une fois, cette propriété n'est plus vérifiée dans le cas d'un parabolique non réduit. Lauritzen, dans [**Lau1**], démontre que, si la caractéristique est strictement supérieure au nombre de Coxeter de G, alors $X = G/P$ est scindée si et seulement si

$$P = {}_mGP_{\mathrm{red}}$$

pour un certain entier m. Autrement dit, si et seulement si X est isomorphe à une variété ayant stabilisateur réduit. Cependant, de notre point de vue cela n'est pas suffisant. En effet, si la caractéristique est plus grande que le nombre de Coxeter, alors tout parabolique est forcément de type standard.

Le théorème suivant, qui est le dernier résultat obtenu dans mon travail de thèse, est un résultat de finitude concernant les propriétés ci-dessus.

Théorème E (Théorème 5.4.18). *Soit $n \geq 1$ un entier fixé.*
Il existe un nombre fini de classes d'isomorphisme de variétés homogènes (projectives, rationnelles) de dimension n dont le fibré anti-canonique est globalement engendré.

Ce résultat a plusieurs conséquences. Pour tout n, il n'existe qu'un nombre fini de classes d'isomorphisme de variétés G/P qui satisfont une des assertions parmi : être de Fano, ou bien être scindée par le Frobenius, ou bien avoir le fibré tangent globalement engendré.

1.6. Organisation du manuscrit

1.6.1. Structure du texte. Le Chapitre 2 est dédié à la motivation du problème, son histoire, les notations et les notions préliminaires nécessaires pour la suite. On y trouve aussi la définition d'isogénie très spéciale et la propriété de factorisation des isogénies énoncée dans Proposition 1.2.1 ; cela correspond à la première partie de l'article [**Mac1**].

Dans le Chapitre 3, nous obtenons la classification de tous les paraboliques ayant une partie réduite maximale (Théorème A et Théorème B), ainsi que l'interprétation géométrique (Théorème C). Cela correspond au coeur de l'article [**Mac1**].

Dans le Chapitre 4, nous obtenons la classification de *tous* les paraboliques (Théorème D), ce qui fait principalement référence à [**Mac2**].

La structure de la dernière partie, c'est-à-dire du Chapitre 5, est un peu plus complexe, car elle mélange plusieurs éléments et résultats de [**Mac1**] ainsi que de [**Mac2**]. D'abord, nous rappelons et précisons un certain nombre de faits sur la décomposition de Białynicki-Birula. Cela nous amène à la construction des diviseurs et des courbes de Schubert qui permettent de décrire le groupe de Picard et le groupe des 1-cycles à équivalence numérique près. Cette description combinatoire constitue un outil très important pour achever la classification du chapitre précédent. Nous passons ensuite aux conséquences géométriques de nos résultats. Tout d'abord, nous présentons de nouveaux exemples de variétés G/P de rang 2, qui ne sont pas de type standard (Proposition 1.5.1). Nous énonçons ensuite des conséquences géométriques plus générales, comme le fait que tout fibré en droites ample soit très ample sur ces variétés. Nous terminons avec un résultat de finitude concernant le fibré anti-canonique des G/P (Théorème E).

Pour des raisons de brièveté, la description concernant le plongement du groupe exceptionnel de type G_2 dans le groupe orthogonal SO_7, ainsi que les calculs des sous-groupes associés aux racines (par rapport à ce plongement) sont regroupés dans le Chapitre 6.

CHAPTER 2

Preliminaries

Abstract. We aim to classify rational projective homogeneous varieties; let us contextualize our problem into a step-by-step framework. First, the question of the relevance of small characteristics is addressed; then we specialize to algebraic groups, introducing infinitesimal groups and related geometric behaviors. Finally, we delve deeper into the classification of non-reduced parabolic subgroups of semisimple groups, together with the history of this mathematical question. This leads to the study of isogenies between simple, simply connected groups.

2.1. Organisation of the manuscript

2.1.1. Structure of the text. The Chapter 2 is dedicated to motivating our classification problem, together with its history and the preliminary notions that are going to be useful later on. We also introduce the notion of very special isogeny, which allows us to state a factorisation property of isogenies, as stated in Proposition 1.2.1. This corresponds to the first part of the article [**Mac1**].

In Chapter 3, we prove the classification of all parabolic subgroups whose reduced part is maximal (Théorème A et Théorème B), as well as the geometric interpretation of such a classification (Théorème C). This corresponds to the hearth of the work in [**Mac1**].

Moving on to Chapter 4, we are able to obtain a statement classifying *all* parabolic subgroup of semisimple groups (Théorème D), which refers mainly to the article [**Mac2**].

The structure of the last part of this manuscript, namely of Chapter 5, is slightly more complex, since it is obtaining by putting together different results coming from [**Mac1**] as well as from [**Mac2**]. First, we recall and precise a certain number of facts concerning the Białynicki-Birula decomposition of homogeneous varieties. This leads us to the construction of the Schubert divisors and the Schubert curves, which allows us to describe the Picard group, as well as the group of 1-cycles up to numerical equivalence. This combinatorial description provides a very important tool in order to achieve the classification of the preceding chapter. Next, we move on to the geometric consequences of our results. First of all, we exhibit new examples of (projective, rational) homogeneous varieties G/P of Picard rank 2, which are not of standard type (by this terminology, we mean that they cannot be expressed as a quotient of a semi-simple group by a parabolic subgroup of standard type; see Proposition 1.5.1). We then move on to prove some more general geometric properties, such as the fact that any ample line bundle on such varieties is actually very ample. The last result presented in this text is a finiteness properties, concerning the anti-canonical line bundle on homogeneous spaces (Théorème E).

For brevity reasons, the description of how the exceptional group of type G_2 embeds into the special orthogonal group SO_7, as well as the computation of the root subgroups (with respect to this embedding) are gathered in Chapter 6.

2.2. Linear algebraic groups in positive characteristics

2.2.1. Small characteristics. The natural question that has been arising in the context of my first three years of research in mathematics is the following: why should an (algebraic) geometer care about problems in small characteristics?

I will try, in a certainly personal and partial way, to give some answers to this question. First of all, when addressing a given classification problem in geometry, we look for a statement which is as concise and as simple as possible, but at the same time which is *uniform*, meaning that it does not depend on the basis over which one is working. Such kind of statements give, in my opinion, a better understanding of the intrinsic geometry of the objects that they describe. A striking example, which is a fundamental result allowing this thesis (and a whole important area of research) to exist, is the classification of split semisimple algebraic groups of adjoint type. These groups can be defined over the ring of the integers, hence they make sense over any basis.

However, the above might not be sufficient by itself as a sole motivation. Another important aspect is that positive characteristics, especially small ones, offer an abundance of solutions for equations. This greater flexibility provides a richer behavior of geometric objects, which can manifest in two fundamentally distinct ways.

Same objects, different properties: this is the case of semisimple algebraic groups over algebraically closed fields. As mentioned above, these objects are the same independently of characteristics; however, they can have nonreduced subgroups in prime characteristics, and the geometric properties of their homogeneous spaces significantly vary, as we explore in a later part of this thesis; see for example [**HL**, Section 4], [**Lau1**] and [**Tot**, Theorem 3.1]. Another significant example of this behavior, which goes beyond the area of group schemes, can be seen in the failure of vanishing theorems in cohomology, for smooth projective varieties.

Entirely new objects: on the other side, entirely new mathematical objects can emerge, some more extravagant than others. The first well-known class of examples in the context of algebraic groups is provided by kernels of the iterated Frobenius homomorphism, which are infinitesimal subgroups arising very naturally in any positive characteristic. Let us also mention the very special isogenies of the simple simply connected groups: these are described below as being exotic homomorphisms with finite kernel defined in characteristic two and three; in the case of characteristic two, their existence is closely related to the behavior of quadratic forms. Aa a last important example, let us mention the classification of simple Lie algebras, which has been completed in characteristic at least five, thanks to the Block–Wilson–Strade–Premet Classification Theorem (see [**Str**]), and which is nowhere near to being finished in characteristic two. Instead, it is still a powerful source of examples and counterexamples, as in [KL], [BLLS] and [CSS].

2.2.2. Basic facts on algebraic groups. From now on, we work over a fixed algebraically closed base field k, of arbitrary characteristic (then rapidly restricting to positive characteristics); unless explicitly mentioned, all objects and maps are supposed to be defined over k. The natural numbers, denoted by $\mathbf{N}$, are assumed to contain zero. Most results we are interested in come from the setting of classical algebraic geometry; nevertheless, we choose to work from the point of view of schemes of finite type over k. For instance, this comes into play when employing terms such as the *kernel* of a homomorphism or the *intersection* of subgroup schemes; these notions are all to be understood in a scheme-theoretic sense. Concerning algebraic groups, our main references are [Bor] and [Spr] for the classical theory over an algebraically closed field; as a more recent monograph adopting the point of view of group schemes, we follow [Mil].

A *k-group scheme* is a scheme over k, together with morphisms

$$m\colon G \times G \longrightarrow G \text{ (multiplication)}, \quad i\colon G \longrightarrow G \text{ (inverse)}$$

and a fixed k-point $e = e_G$, the neutral element, satisfying the usual group axioms. In other words, it is a group object in the category of schemes over k.

An *algebraic group* is a group scheme whose underlying scheme is of finite type over k. By subgroup, we always mean a closed subgroup scheme; this is a crucial point, since we sometimes deal with infinitesimal subgroups. When defining a certain object, such as a scheme or a subgroup, we often make use of the functorial notation, identifying the object with its functor of points: namely, a group scheme is given by a representable functor from the category of schemes over k (or of k-algebras) to the category of groups.

For instance, the additive group $\mathbf{G}_a$, the multiplicative group $\mathbf{G}_m$ and the special linear group SL_n (over k) have as underlying schemes smooth, affine k-varieties, and represent respectively the following functors from the category of k-algebras to that of groups:

$$R \longmapsto (R, +), \quad R \longmapsto (R^\times, \cdot), \quad R \longmapsto \mathrm{SL}_n(R).$$

We often omit to indicate the variable R, ranging over the set of all k-algebras, and write for example

$$G = \{(x, y) \in \mathbf{G}_a^2 \colon x^p = x + y^p\} \subset \mathbf{G}_a^2.$$

In another direction, a standard example of algebraic group is given by elliptic curves; being projective varieties, their functor of points is not so easy to define as in the three cases above.

Definition 2.2.1. An algebraic group is said to be *linear* if its underlying variety is affine. Equivalently, it admits a closed embedding into some general linear group GL_N for some natural number N.

The study of algebraic groups is usually divided in two, essentially orthogonal, parts: linear groups and Abelian varieties. The latter are defined as being smooth, connected projective algebraic groups (which in particular implies commutativity of the group structure), and can be thought of the higher-dimensional generalisation of elliptic curves. This *dévissage* into two families can be obtained thanks to the following structure theorem of Chevalley, proven by Barsotti and Rosenlicht (see [**Ros**]).

THEOREM 2.2.2. *Let G be a smooth connected algebraic group; then there is a unique maximal smooth connected linear algebraic subgroup G_{aff}, which is normal. Moreover, the quotient G/G_{aff} is an Abelian variety.*

In this thesis, we focus on linear algebraic groups and in particular on their actions on smooth projective varieties; we soon further restrict to a very specific and *well behaved* class, that of semisimple algebraic groups.

2.2.3. Frobenius kernels. By a fundamental result due to Cartier, all algebraic groups over an algebraically closed field k of characteristic zero are *smooth* (or equivalently, reduced) over k. However, this very strong property does not hold in positive characteristic. In this section, we work in characteristic $p > 0$ and introduce a universal class of examples of non-reduced algebraic groups: Frobenius kernels. Let

$$F = F_k : k \longrightarrow k, \quad t \longmapsto t^p$$

be the Frobenius homomorphism of the base field k and let A be a k-algebra. We define $A^{(1)}$ as the tensor product below.

$$\begin{array}{ccc} A^{(1)} = A \otimes_{k,F} k & \longleftarrow & A \\ \uparrow & & \uparrow \\ k & \xleftarrow[F]{} & k \end{array}$$

In other words, the k-algebra structure of $A^{(1)}$ is given by

$$t \cdot a = t^p a, \quad \text{for all } t \in k,\ a \in A^{(1)}.$$

The (relative) *Frobenius morphism* of a k-algebra A is the k-algebra homomorphism

$$F_A : A^{(1)} \longrightarrow A, \quad a \otimes t \longmapsto ta^p,$$

obtained by the universal property of the tensor product.

Next, let $X = \operatorname{Spec} A$ be an affine scheme and let

$$X^{(1)} := \operatorname{Spec} A^{(1)}$$

be its base change with respect to the Frobenius morphism. The k-scheme morphism associated to F_A is called the (relative) *Frobenius morphism* of X and is denoted as

$$F_X : X \longrightarrow X^{(1)}.$$

In other words, the Frobenius morphism of a scheme X can be thought of as being the identity on the underlying topological space of X, while it raises functions to their p-th power. The fiber product above is needed in order to make F_X a morphism of k-schemes, which would not be the case otherwise because the Frobenius is clearly not a k-linear map. By repeating this construction, we define, for any k-scheme X and for any natural number $m \geq 1$, its m-th Frobenius twist $X^{(m)}$ and its m-th *iterated* Frobenius morphism

$$F_X^m \colon X \longrightarrow X^{(m)}.$$

Proposition 2.2.3. *Let $G = \operatorname{Spec} A$ be a linear algebraic group. Then $G^{(1)}$ is also a linear algebraic group and the iterated Frobenius morphism F_G^m is a morphism of algebraic groups for any m. We denote the* Frobenius kernel *of G as*

$$_mG := \ker(F_G^m \colon G \longrightarrow G^{(m)}).$$

A direct consequence of their definition is that Frobenius kernels are *infinitesimal* subgroups, meaning that their underlying topological space is a point. This is a basic remark to make, but it will be essential in order to understand the structure of parabolic subgroups in positive characteristics.

Example 2.2.4. The Frobenius kernel of the additive group is denoted as $\boldsymbol{\alpha}_p \subset \mathbf{G}_a$; the Frobenius kernel of the multiplicative group as $\boldsymbol{\mu}_p \subset \mathbf{G}_m$. Moving on to $G = \mathrm{SL}_2$, we get

$$_1G = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2 \colon a^p = d^p = 1,\ b^p = c^p = 0 \right\}.$$

Definition 2.2.5. Let G be a linear connected algebraic group and H be a nontrivial subgroup. The *height* of H (when it is defined) is the smallest positive integer m such that H is killed by the m-th iterated Frobenius morphism; equivalently, such that $H \subset {}_mG$. A subgroup H has finite height if and only if it is infinitesimal.

2.2.4. Restricted Lie algebras. Let us gather here some basic facts about Lie algebras of algebraic groups in positive characteristics; the main references for this subject are [**Hum**] and [**Str**].

A *Lie algebra* is a k-vector space L, endowed with an operation

$$L \times L \longrightarrow L, \quad (u, v) \longmapsto [u, v]$$

called the *bracket* operation (or the *commutator*) and satisfying the following axioms:
- the bracket operation is bilinear;
- $[u, u] = 0$, for all $u \in L$;
- $[u, [v, w]] + [v, [w, u]] + [w, [u, v]] = 0$ for all $u, v, w \in L$. The last condition is called the *Jacobi identity*. Homomorphisms of Lie algebras are linear maps respecting the bracket operation; Lie subalgebras and Lie ideals are defined in the analogous way. For a fixed element v of a Lie algebra L, its *adjoint map* is defined as

$$\operatorname{ad}(v) \colon L \longrightarrow L, \quad u \longmapsto [v, u];$$

this defines a morphism ad from L to the space of its k-linear derivations, called the *adjoint representation* of L.

One of the equivalent ways to associate to an algebraic group G its Lie algebra $\operatorname{Lie} G$ (and the most concrete) is to consider the tangent space at the identity element;

$$\operatorname{Lie} G := \operatorname{Hom}_k(I/I^2, k) = T_eG,$$

where $e \in G(k)$ denotes the identity of G and I the maximal ideal of the local ring of G at e. Concerning the bracket operation, it can be thought of geometrically as arising from the interpretation of $\operatorname{Lie} G$ as the algebra of left invariant derivations on G.

Due to the smoothness of algebraic groups, in characteristic zero we can recover a lot of information about a subgroup from its Lie algebra; namely, if H is a subgroup of G, both are connected and have the same Lie algebra, then they must coincide. In positive characteristic, this is clearly not enough, because the Lie algebra of a connected linear algebraic group coincides with the one of its Frobenius kernel. In order to deal with this issue, we need to introduce more structure.

Definition 2.2.6. A *p-mapping* on a Lie algebra L is a map

$$(-)^{[p]} \colon L \longrightarrow L$$

such that
- for all $v \in L$, $\operatorname{ad}(v^{[p]}) = (\operatorname{ad}(v))^p$;
- for all $\lambda \in k$ and all $v \in L$, $(\lambda v)^{[p]} = \lambda^p v^{[p]}$;
- for all $u, v \in L$, we have

$$(u+v)^{[p]} = u^{[p]} + v^{[p]} + \sum_{i=1}^{p-1} s_i(u,v),$$

where the s_i are defined as follows:

$$s_i(u,v) := -\frac{1}{i} \sum_{\sigma} \operatorname{ad}(\sigma(1)) \operatorname{ad}(\sigma(2)) \cdots \operatorname{ad}(\sigma(p-1))(v),$$

with σ running over the maps from $\{1, \ldots, p-1\}$ to $\{u, v\}$ taking exactly i times the value u.

Essentially, the above three conditions generalise the notion of the Frobenius morphism; namely, any associative k-algebra A can be endowed with a natural Lie algebra structure and a p-mapping, given by

$$[a,b] = ab - ba \quad \text{and} \quad a \longmapsto a^p$$

respectively.

Definition 2.2.7. A *p-Lie algebra*, or *restricted* Lie algebra, is a Lie algebra equipped with a p-mapping.

The following is a crucial result (see [**SGA3**, Exposé VVIA, section 7]) which helps us see clearly how much information we can recover from the Lie algebra of a subgroup; it is frequently employed throughout this thesis.

THEOREM 2.2.8. *Let G be an algebraic group. There is an equivalence of categories between subgroups of G of height one and p-Lie subalgebras of* $\operatorname{Lie} G$*, explicitely given by* $H \rightsquigarrow \operatorname{Lie} H$*.*

2.3. Parabolic subgroups and homogeneous varieties

We introduce here our main object of study, namely homogeneous varieties: we start by a clear definition of what this means, then move on to a structure theorem, illustrating why our classification problems can be reduced to the study of parabolic subgroup schemes of semisimple groups. Such subgroups are well understood in characteristic zero, but many more appear in positive characteristics, as they can be non-reduced. We start by introducing their combinatorial description in characteristic zero, then move on to the work of [**Wen**], [**HL**] in characteristic at least five, which finally leads to the start of my PhD.

Remark 2.3.1. The definition of projective homogeneous spaces that is adopted here relies on the notion of automorphism group; for the sake of clarity, let us recall it. For a proper algebraic scheme X over a perfect field k, the functor

$$\underline{\mathrm{Aut}}_X \colon (\mathbf{Sch}/k)_{\mathrm{red}} \longrightarrow \mathbf{Grp}, \quad T \longmapsto \mathrm{Aut}_T(X_T),$$

sending a reduced k-scheme T to the group of automorphisms of T-schemes of $X \times_k T$, is represented by a reduced group scheme $\underline{\mathrm{Aut}}_X$ which is locally of finite type over k: see [**MO**, Theorem 3.6]. We denote as

$$\underline{\mathrm{Aut}}_X^0$$

its identity component, which is a smooth connected algebraic group, acting faithfully on X.

Definition 2.3.2. A projective variety X is said to be *homogeneous* if the algebraic group $\underline{\mathrm{Aut}}_X^0$ acts transitively on the underlying topological space of X. In particular, such varieties are smooth.

For instance, for any natural number $n \geq 1$, the projective space $\mathbf{P}^n$ is homogeneous under the action of its automorphism group PGL_{n+1}.

2.3.1. Semisimple groups and root systems.

Let us introduce the notions of reductive and semisimple groups and illustrate the reason why we can restrict to these classes of groups for our purposes.

Definition 2.3.3. A *torus* is an algebraic group T, isomorphic to some power $\mathbf{G}_m^r$ of the multiplicative group. A *Borel subgroup* of a smooth linear algebraic group G is a maximal smooth connected solvable subgroup. A *parabolic subgroup* is a subgroup P of G such that the quotient G/P is projective.

Proposition 2.3.4. *Let G be a smooth linear algebraic group. The Borel subgroups of G are all $G(k)$-conjugate; these are characterized by being the smooth connected solvable subgroups B such that the quotient G/B is projective. Moreover, parabolic subgroups are precisely those containing a Borel subgroup.*

Definition 2.3.5. The *unipotent radical* $R_u(G)$ of an algebraic group G is the largest smooth connected unipotent normal subgroup. A *reductive* group is a smooth, connected, linear algebraic group G such that $R_u(G) = e$; equivalently, it does not contain any nontrivial smooth connected unipotent normal subgroup.

Let us emphasize that the above does not coincide with the notion of linear reductivity in general; a group is said to be *linearly reductive* if all of its representations are semisimple. This holds true for reductive groups in characteristic zero, however, it fails in positive characteristic. Indeed, consider the group GL_n over a field of characteristic $p > 0$; together with its action on

$$V := \mathrm{Sym}^p(k^n),$$

coming from the standard action on k^n via base change. Then V is a G-module containing a simple submodule, consisting of the elements v^p with $v\ inV$, which does not admit any stable complement.

Definition 2.3.6. The *radical* $R(G)$ of an algebraic group G is the largest smooth connected solvable normal subgroup. A *semisimple* group is a smooth, connected, linear algebraic group G such that $R(G) = e$; equivalently, it does not contain any nontrivial connected solvable normal subgroup.

The following fundamental structure result is due to [**Sal**, Theorem 5.2].

THEOREM 2.3.7. *Let X be a smooth projective homogeneous variety; then there is an isomorphism*

$$X \simeq A \times G/P,$$

where A is an Abelian variety, G is a semisimple group of adjoint type and P a parabolic subgroup of G.

Let us notice that, since the center of G is contained in any maximal torus, we can replace G with its simply connected cover and P with the corresponding pre-image. Moreover, the above structure theorem implies that all homogeneous varieties with discrete Picard group are of the form G/P, because the Abelian variety in the statement must be reduced to a point.

Remark 2.3.8. Let us make some historical observations concerning terminology: the term *flag variety* associated to a semisimple group G denotes the smooth projective variety G/B where B is a Borel subgroup. This choice comes from the fact that, for the special linear group $G = \mathrm{SL}_n$, the flag variety is the space parametrizing all complete flags (of vector subspaces) of an n-dimensional vector space over k. A natural generalization is the notion of *generalized flag variety*, which is usually employed when speaking about the variety G/P, where G is semisimple and P is some reduced parabolic subgroup. Since here we are mostly interested in dealing with non-reduced parabolic subgroups, it is worth mentioning that [**HL**] choose the term *unseparated flag variety* when P is non-reduced. In this text, we prefer to call all of the above spaces just *rational projective homogeneous* varieties. These varieties are rational because they contain an open subset isomorphic to affine space; this can be seen for example by taking the open cell with respect to the Białynicki-Birula decomposition; this decomposition is discussed in detail in Chapter 5.

Now that we have made the reduction to the semisimple case, we can introduce some standard important notation, which allows to define the very important ingredient of the structure of the semisimple groups: root systems.

Let us consider, for any torus $S \simeq \mathbf{G}_m^r$, the free Abelian groups

$$X(S) = \mathrm{Hom}_{\mathrm{Grp}}(S, \mathbf{G}_m) \quad \text{and} \quad X_*(S) = \mathrm{Hom}_{\mathrm{Grp}}(\mathbf{G}_m, S);$$

these are called respectively the group of *characters* and *cocharacters* of S and are isomorphic to $\mathbf{Z}^r$. For historical reasons via the theory of compact Lie groups, it is a standard convention to use additive notation when discussing elements of the character and cocharacter lattices. Composition defines the following perfect duality

$$X(S) \times X_*(S) \longrightarrow \mathbf{Z}, \quad (\alpha, \lambda) \longmapsto \alpha \circ \lambda.$$

A very important property of tori (and more generally, of their subgroups, which are called groups of multiplicative type) is that their representations are completely reducible. More precisely, any finite-dimensional linear representation V of a torus S decomposes as a direct sum

$$V = \bigoplus_{\lambda \in X(S)} V_\lambda, \quad V_\lambda = \{v \in V \colon t \cdot v = \lambda(t) v, \text{ for all } t \in S\}.$$

Using terminology coming from representation theory, we call V_λ the *weight space* associated to the character λ; the finitely many characters with nonzero weight space are called *weights* of the S-action on V.

Next, let us go back to our maximal torus T contained in the semisimple group G, and consider the T-action on the group G given by conjugation; this yields a T-action on the Lie algebra $\mathrm{Lie}\, G$. By the complete reducibility property we just stated, there is a decomposition into weight spaces

$$\mathrm{Lie}\, G = \mathrm{Lie}\, T \oplus \left(\bigoplus_{\gamma \in \Phi} \mathfrak{g}_\gamma \right),$$

where (by another impressive property of groups of multiplicative type - see [CGP, Proposition A.8.10] or [Mil, Theorem 13.33]) the fixed points of the action coincide with $\mathrm{Lie}\, T$. The finite set

$$\Phi = \Phi(G, T)$$

consists of the non-trivial torus characters such that the corresponding weight space

$$\mathfrak{g}_\gamma := \{X \in \mathrm{Lie}\, G \colon t \cdot X = \gamma(t) X, \text{ for all } t \in T\}$$

is nonzero. These are called the *roots* of G with respect to the maximal torus T. It is an important structure result that the root spaces $\mathfrak{g}_\gamma$ are actually one dimensional, and lift to copies of the additive group inside of G; more precisely, there are T-equivariant isomorphisms

$$u_\gamma \colon \mathbf{G}_a \xrightarrow{\sim} U_\gamma \subset G,$$

so that the T-action is given by

$$t \cdot u_\gamma(x) = u_\gamma(\gamma(t) x) \text{ for all } t \in T,$$

where $\mathbf{G}_m$ acts on $\mathbf{G}_a$ by scalar multiplication. The subgroup U_γ is called a *root subgroup*, and the corresponding map u_γ is the associated *root homomorphism*. We denote as Φ^+ the subset of positive roots associated to the fixed Borel subgroup B; namely, a root γ is positive if and only if the root subgroup U_γ is contained in B. Moreover, once the Borel subgroup is chosen, there is a uniquely determined *basis* of the so-called *simple* roots, namely a subset $\Delta \subset \Phi^+$ such that any positive (resp. negative) root can be written as a linear combination of elements of Δ, with non-negative (resp. non-positive) integer coefficients. The group G is generated by the maximal torus, together with the root subgroups associated to simple roots and to their opposites.

Example 2.3.9. As a universal fundamental example, let us consider $G = \mathrm{SL}_n$, together with the maximal torus T of diagonal matrices

$$T \ni t = \mathrm{diag}(t_1, \dots, t_n), \quad t_1 \cdot \ldots \cdot t_n = 1$$

and the Borel subgroup B of upper triangular invertible ones. Then the weight decomposition of the Lie algebra of G is as follows:

$$\mathrm{Lie}\, G = \mathrm{Lie}\, T \oplus \left(\bigoplus_{i \neq j} k E_{ij} \right),$$

where the matrix E_{ij} has all zero entries except for a 1 in the (i, j)-th position. Denoting as ε_i the character sending t to t_i, one has that T acts on the line kE_{ij} with weight $\varepsilon_i - \varepsilon_j$, hence

$$\Phi = \{\varepsilon_i - \varepsilon_j, i \neq j\} \supset \Phi^+ = \{\varepsilon_i - \varepsilon_j, i < j\} \supset \Delta = \{\varepsilon_i - \varepsilon_{i+1}, 1 \leq i \leq n-1\}.$$

For more details on this, including the abstract definitions of root system, root datum, the Existence and Isogeny Theorem for semisimple groups and the corresponding Dynkin diagrams, we refer to [**Bou**] and to [**Mil**, Appendix C].

2.3.2. Reduced parabolics. Let $G \supset B \supset T$ be respectively a semisimple, simply connected algebraic group over k, a Borel subgroup and a maximal torus contained in it. Our aim is to classify all homogeneous projective G-varieties; these are quotients of the form G/P, where P is a parabolic subgroup of G. By the conjugacy of the Borel subgroups (Proposition 2.3.4) we might restrict ourselves to those containing the Borel subgroup B; from now on, every parabolic subgroup satisfies this assumption, unless otherwise mentioned.

Definition 2.3.10. For any simple root α, there is a unique maximal (reduced) parabolic subgroup generated by B and by all the $U_{-\beta}$, where $\beta \neq \alpha$ ranges over the simple roots. It is uniquely determined as being the largest reduced subgroup containing B and not containing the root subgroup $U_{-\alpha}$. We will denote it as

$$P^\alpha;$$

let us emphasise that the above notation is not standard and that these subgroups play a crucial role in our classification problem.

Example 2.3.11. Let us keep the notation of Example 2.3.9. As a parabolic subgroup of SL_n, one can consider the following:

$$\left\{\begin{pmatrix} * & & & & \\ & * & & & \\ & & \ddots & & \\ & & & * & \\ & & & & * \end{pmatrix}\right\} \subset \left\{\begin{pmatrix} * & & \cdots & & * \\ & * & & & \\ & & \ddots & & \vdots \\ & & & * & \\ & & & & * \end{pmatrix}\right\} \subset P_m := \left\{\begin{pmatrix} * & * & \cdots & & * \\ * & * & \cdots & & * \\ & & * & \cdots & * \\ & & \vdots & \ddots & \vdots \\ & & * & \cdots & * \end{pmatrix}\right\},$$

In other words, the subgroup P_m, for $1 \leq m \leq n-1$, is generated by the Borel subgroup together with two blocks of size m and $n-m$ on the diagonal. In more geometric terms, we can observe that the corresponding homogeneous variety G/P_m is the Grassmannian of m-dimensional vector subspaces in k^n. With the notation introduced in Definition 2.3.10,

$$P_m = P^{\varepsilon_m - \varepsilon_{m+1}};$$

indeed, P_m contains all the copies of $\mathbf{G}_a$ corresponding to entries just below the diagonal, but it has trivial intersection with $U_{-\varepsilon_m+\varepsilon_{m+1}}$.

More generally, we can classify in a combinatorial way all *reduced* parabolic subgroups of a given semisimple group: once the Borel is fixed, these subgroups are uniquely determined by the simple roots forming a basis for the root system of a Levi subgroup. This is a well-known result; let us formulate it in a slightly different way, focusing rather on the simple roots which are *not* contained in a Levi subgroup. We make this choice of notation because it makes the statement easier to generalize and adapt to the non-reduced case later on.

THEOREM 2.3.12. *There is a bijection between subsets of the basis Δ and reduced parabolic subgroups, associating to some $I \subset \Delta$ the subgroup*

$$P_I = \bigcap_{\alpha \in \Delta \setminus I} P^{\alpha}.$$

In particular, the above result allows one to classify all parabolic subgroups over an algebraically closed field *of characteristic zero.*

2.3.3. The work of Wenzel, Haboush and Lauritzen. From now on, unless otherwise stated, we will place ourselves over an algebraically closed field k of prime characteristic $p > 0$. Let us start with a few examples, which are rather easy to construct but contain already a lot of geometric information.

Example 2.3.13. Let $G = \mathrm{SL}_2$; the simplest non-reduced parabolic subgroup one can construct is

$$P = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2 \colon c^p = 0\right\} = {}_1GB \subset G,$$

which is obtained from the Borel by fattening with the Frobenius kernel (see Example 2.2.4 above). The corresponding homogeneous variety, however, is nothing new: it is the projective line $\mathbf{P}^1$, together with the standard SL_2-action twisted once by the Frobenius morphism.

Example 2.3.14. Let us consider the following twisted incidence variety in the product of two projective planes

$$\mathrm{SL}_3 \circlearrowright X_m := \{x_0^{p^m} y_0 + x_1^{p^m} y_1 + x_2^{p^m} y_2 = 0\} \subset \mathbf{P}^2 \times \mathbf{P}^2,$$

where m is a non-negative integer. Moreover, let us consider the action of $G = \mathrm{SL}_3$ on $\mathbf{P}^2 \times \mathbf{P}^2$, acting on the first copy of $\mathbf{P}^2$ with the standard action by base change, and on the second copy of $\mathbf{P}^2$ with the linear dual of the same action, but twisted by an m-th Frobenius morphism. This way, the variety X_m is preserved by the action, and we can see it as an SL_3-homogeneous variety. If we consider the coordinate point

$$([1:0:0],[0:0:1]) \in X_m,$$

an explicit computation of its stabiliser $P_{(m)}$ gives

$$P_{(m)} = \left\{ \begin{pmatrix} a & b & c \\ 0 & e & f \\ 0 & h & i \end{pmatrix} \in \mathrm{SL}_3 \colon\ h^{p^m} = 0 \right\} \subset \mathrm{SL}_3 .$$

Let us notice that for $m = 0$, the parabolic subgroup $P_{(0)}$ is just the Borel subgroup of upper triangular matrices in G; on the other hand, for any $m \geq 1$ we have that $P_{(m)}$ is a *non-reduced* parabolic.

2.3.3.1. *Results in any characteristic.* Let us review the main results which were known on the structure of non-reduced parabolic subgroups, focusing on the statements which hold without any assumption on the characteristic.

For a non-reduced parabolic subgroup P with reduced part P_{red}, let

$$U_P^- := P \cap R_u(P_{\mathrm{red}}^-) \tag{2.3.1}$$

be its intersection with the unipotent radical of the opposite reduced parabolic of P_{red}. The subgroup U_P^- is unipotent, infinitesimal and satisfies

$$U_P^- = \prod_{\gamma \in \Phi^+ \setminus \Phi_I} (U_P^- \cap U_{-\gamma}) \quad \text{and} \quad P = U_P^- \times P_{\mathrm{red}}, \tag{2.3.2}$$

where both identities are isomorphisms of schemes given by the multiplication of G. This implies that P can be recovered from its reduced part P_{red}, together with its intersections with all the root subgroups contained in the opposite unipotent radical $R_u(P_{\mathrm{red}}^-)$. Let us reformulate this statement in a more combinatorial fashion, introducing a numerical function. We denote the kernel of the n-th iterated Frobenius of the additive group $\mathbf{G}_a$ as $\boldsymbol{\alpha}_{p^n}$; while $\boldsymbol{\alpha}_{p^\infty}$ is understood to be $\mathbf{G}_a$.

Definition 2.3.15. Let P be a parabolic subgroup of a semisimple group G. The *associated function*

$$\varphi \colon \Phi \longrightarrow \mathbf{N} \cup \{\infty\}$$

is given by the identity

$$P \cap U_{-\gamma} = u_{-\gamma}(\boldsymbol{\alpha}_{p^{\varphi(\gamma)}}), \quad \gamma \in \Phi^+.$$

In other words, any positive root γ not belonging to the root system of the Levi subgroup $P_{\text{red}} \cap P^-_{\text{red}}$ is sent to the natural number corresponding to the height of the finite unipotent subgroup $P \cap U_{-\gamma}$; while all other roots are sent to infinity. For instance, the associated function to the parabolic ${}_mGP^\alpha$ sends all positive roots to infinity, except for those containing α in their support, which assume value m. The fundamental structure result below is [**Wen**, Theorem 10].

THEOREM 2.3.16. *The parabolic subgroup P is uniquely determined by the function φ, with no assumption on the characteristic or on the Dynkin diagram of G.*

2.3.3.2. *Classification in characteristic at least five.* Let us assume that the characteristic of the base field is at least 5, or that the Dynkin diagram of G is simply laced (namely, that each simple factor of G is of type A_n, D_n, E_6, E_7 or E_8). The (partial) classification is illustrated in [**Wen**] and [**HL**] with the following statement.

THEOREM 2.3.17. *Under the above assumption, the associated function φ is uniquely determined by its restriction to Δ.*

Let us introduce a particular class of parabolic subgroups, which allows us to reformulate the above Theorem in a simpler way, slightly generalizing the formulation of Theorem 2.3.12.

Definition 2.3.18. A parabolic subgroup is said to be *of standard type* if it is of the form

$$P = \bigcap_{\alpha \in \Delta \setminus I} {}_{m_\alpha}GP^\alpha$$

for some non-negative integers m_α.

Let us notice that, for a parabolic subgroup of standard type, the integers m_α are uniquely determined as being the height of the intersection

$$U_{-\alpha} \cap P.$$

For instance, the parabolic subgroup $P_{(m)}$ of Example 2.3.14 above is of standard type: let α_1 and α_2 be the simple roots of SL_3 with respect to the Borel subgroup of upper triangular matrices, namely

$$\alpha_1 \colon \mathrm{diag}(t_1, t_2, t_3) \mapsto t_1/t_2 \quad \text{and} \quad \alpha_2 \colon \mathrm{diag}(t_1, t_2, t_3) \mapsto t_2/t_3.$$

Then we can write

$$P_{(m)} = P^{\alpha_1} \cap {}_mGP^{\alpha_2}.$$

Corollary 2.3.19. *If $p \geq 5$ or G is simply laced, all parabolic subgroups are of standard type.*

In other words, all parabolic subgroups of G can be obtained from reduced ones by fattening with Frobenius kernels and intersecting. More precisely, they are all of the form

$${}_{m_1}GP^{\beta_1} \cap \ldots \cap {}_{m_r}GP^{\beta_r},$$

for some simple roots $\beta_1, \ldots, \beta_r$ and some non-negative integers $m_1, \ldots, m_r$.

The proof of [**Wen**] relies heavily on the structure constants (defined over $\mathbf{Z}$) relative to a Chevalley basis of the Lie algebra of a simply connected semisimple group. By construction such constants are integers with absolute value strictly less than five: the hypothesis on the characteristic and on the Dynkin diagram guarantees that they do not vanish over k. This leads to the following natural question, which marks the starting point of my PhD thesis:

How can we extend the above results to characteristic two and three?

The above problem has been open since 1993, because [**Wen**] makes only an allusion to the fact that there exist examples of exotic (meaning not of standard type) parabolic subgroups in type B_2 and G_2. The best result we could hope for was a uniform description of parabolics, independent of characteristics; this was the guiding philosophy behind this work. Actually, due to the vanishing of structure constants, some exotic objects in characteristic two and type G_2 arise anyway. This is justified and described in the following chapter.

2.4. Notation and conventions

Let k be an algebraically closed field of prime characteristic $p > 0$; when V is a finite-dimensional k-vector space, we adopt the convention for the projective space $\mathbf{P}(V)$ to be lines in V. Unless otherwise stated, all objects and morphisms are defined over k. We work in the setting of group schemes of finite type over k: the terms homomorphism, kernel, subgroup, intersection are to be understood in such a setting. For two subgroups H and K of G, we denote as

$$\langle H, K \rangle$$

the smallest subgroup of G containing both of them.
We consider semisimple groups G, together with a Borel subgroup B and a maximal torus T contained in it. Let us call Φ the root system of G relative to T, Φ^+ the subset of positive roots associated to the Borel subgroup B and Δ the corresponding basis of simple roots. The corresponding opposite Borel subgroup is denoted as B^-. Let

$$W = W(G, T) = N_G(T)/T$$

be the Weyl group of G relative to T; moreover, let $s_\alpha \in W$ be the reflection associated to the simple root $\alpha \in \Delta$.
For any root γ, let $\mathrm{Supp}(\gamma)$ be its *support*, defined as being the set of simple roots which have a nonzero coefficient in the expression of γ as linear combination of simple roots with integer coefficients of the same sign. Moreover, we denote as U_γ the root subgroup associated to γ, as $u_\gamma \colon \mathbf{G}_a \to U_\gamma$ its root homomorphism and as $\mathfrak{g}_\gamma$ the corresponding one-dimensional Lie subalgebra of $\mathrm{Lie}\, G$. When G is simply connected, let us fix a Chevalley basis of $\mathrm{Lie}\, G$:

$$\{X_\gamma, H_\alpha \colon \ \gamma \in \Phi, \alpha \in \Delta\}.$$

In particular,

$$\mathfrak{g}_\gamma = \mathrm{Lie}\, U_\gamma = kX_\gamma \quad \text{and} \quad X_\gamma = \mathrm{d}u_\gamma(1).$$

As in Theorem 2.3.12 above, we denote as P_I the reduced parabolic subgroup having $I \subset \Delta$ as set of simple roots of a Levi subgroup. Moreover, we write ${}_mG$ for the kernel of the m-th iterated Frobenius morphism

$$F_G^m : G \to G^{(m)},$$

while the maximal reduced parabolic subgroup not containing $U_{-\alpha}$, for α a simple root, is called

$$P^\alpha := P_{\Delta \setminus \{\alpha\}}.$$

Concerning irreducible root systems of the different Dynkin types and their respective basis, we follow notations from [**Bou**].

2.5. Isogenies with no central factor

The guiding idea is to mimic the known classification, illustrated in Theorem 2.3.12 and in Corollary 2.3.19 in terms of reduced parabolics and Frobenius kernels, by replacing the Frobenius morphism with any noncentral isogeny (see Theorem 3.3.2). This motivates the preliminary study and classification of such homomorphisms.

2.5.1. Classifying isogenies. We now classify isogenies between simple algebraic groups, first recalling definitions and the Isogeny Theorem, then introducing the so-called *very special isogeny* π_G, whose kernel is a certain subgroup of height one defined by short roots - which only exists when the Dynkin diagram has an edge of multiplicity equal to the characteristic - and concluding with the following factorisation result: see Proposition 2.5.12.

Proposition 2.5.1. *Let G be a simple and simply connected algebraic group over k. Let $f : G \to G'$ be an isogeny. Then there exists a factorisation of f as*

$$f : G \xrightarrow{\quad \pi \quad} \overline{G} \xrightarrow{\quad F_{\overline{G}}^m \quad} (\overline{G})^{(m)} \xrightarrow{\quad \rho \quad} G',$$

where m is a natural number, ρ is a central isogeny and π is either the identity or - when the Dynkin diagram of G has an edge of multiplicity p - the very special isogeny π_G.

We shall start by reviewing what isogenies look like, in particular noncentral ones. First, let us recall some notations and the statement of the Isogeny Theorem, which is proved in detail in [**Ste**].

Definition 2.5.2. Let (G, T) and (G', T') be reductive algebraic groups over k, equipped with maximal tori. An *isogeny* between them is a surjective homomorphism of algebraic groups $f : G \to G'$ having finite kernel, sending the maximal torus T to the maximal torus T'. The *degree* of f is the order of its kernel. An isogeny is called *central* if its kernel is contained in the center of G.

Given an isogeny f, there is an induced map between the character groups

$$\varphi := X(f_{|T}) : X(T') \longrightarrow X(T), \quad \chi' \longmapsto \chi' \circ f_{|T},$$

satisfying the conditions :

(i) both $\varphi : X(T') \to X(T)$ and its dual $\varphi^\vee : X^\vee(T) \to X^\vee(T')$ are injective,

(ii) there exists a bijection $\Phi \leftrightarrow \Phi'$, denoted $\alpha \leftrightarrow \alpha'$, and integers $q(\alpha)$ which are all powers of p, such that

$$\varphi(\alpha') = q(\alpha)\alpha \quad \text{and} \quad \varphi^\vee(\alpha^\vee) = q(\alpha)\alpha^{\vee\prime} \quad \text{for all } \alpha \in \Phi.$$

Geometrically, the integers $q(\alpha)$ arise as follows: the image $f(U_\alpha)$ is a smooth connected unipotent algebraic subgroup of G' which is normalized by T' and isomorphic to the additive group $\mathbf{G}_a$, hence it must be of the form $U_{\alpha'}$ for a unique $\alpha' \in \Phi'$. This gives the bijection; then, using the T-action on those two root subgroups, one finds that there exists a constant $c_\alpha \in \mathbf{G}_m$ and an integer $q(\alpha) \in p^{\mathbf{N}}$ such that

$$f(u_\alpha(x)) = u_{\alpha'}(c_\alpha x^{q(\alpha)}) \tag{2.5.1}$$

for all $x \in \mathbf{G}_a$.

Definition 2.5.3. A homomorphism between character groups $\varphi\colon X(T') \to X(T)$ satisfying conditions (i) and (ii) is called an *isogeny of root data.*

THEOREM 2.5.4 (*Isogeny Theorem*). *Let (G,T) and (G',T') be reductive algebraic groups over k. Assume given an isogeny of root data $\varphi\colon X(T') \to X(T)$. Then there exists an isogeny $f\colon (G,T) \to (G',T')$ inducing φ. Moreover, f is unique up to an inner automorphism* inn(t) *for some $t \in (T'/Z(G'))(k)$.*

PROOF. See [**Ste**, 1.5]. □

For instance, an important class of isogenies is given by the ones having central kernel, which are characterized by the fact that the associated integers $q(\alpha)$ are all equal to 1: these are not interesting for our purpose of studying parabolic subgroups, since we may restrict ourselves in the classification to the case of a simply connected group (or an adjoint one, depending on the desired properties). The most known example of a noncentral isogeny is an iterated Frobenius homomorphism F^m, for which $\alpha' = \alpha$ and all $q(\alpha)$ are equal to p^m. Do other isogenies exist? We shall now consider this question.

2.5.1.1. *Very special isogenies.* Let us make, for the remaining part of this Chapter, the assumption that G is simple. The Weyl group $W = W(G,T)$ acts on roots leaving the integer q invariant: if the Dynkin diagram of G is simply laced, then there is only one orbit, hence all $q(\alpha)$ must assume the same value. This means, by the Isogeny Theorem, that up to inner automorphisms the only noncentral isogenies with source G are iterated Frobenius homomorphisms.

On the other hand, assume that the Dynkin diagram of G has a multiple edge. In this setting, there are two distinct orbits under the action of the Weyl group, corresponding to long and short roots: this allows us, considering an isogeny $f\colon (G,T) \longrightarrow (G',T')$, for two possibly distinct values of $q(\alpha)$. Let us denote as $\Phi_<$ and $\Phi_>$ the subsets of Φ consisting of short and long roots respectively, and denote the two integer values as

$$q_< := q(\alpha)\,(\alpha \in \Phi_<) \quad \text{and} \quad q_> := q(\alpha)\,(\alpha \in \Phi_>). \tag{2.5.2}$$

Analogously, we fix the following notation for the direct sum of root spaces associated to roots of a fixed length:

$$\mathfrak{g}_< := \bigoplus_{\alpha \in \Phi_<} \mathfrak{g}_\alpha = \bigoplus_{\alpha \in \Phi_<} \operatorname{Lie} U_\alpha \quad \text{and} \quad \mathfrak{g}_> := \bigoplus_{\alpha \in \Phi_>} \mathfrak{g}_\alpha = \bigoplus_{\alpha \in \Phi_>} \operatorname{Lie} U_\alpha.$$

We now recall a notion introduced in [CGP, Section 7.1], based on previous work from Borel and Tits, and some of its properties. Also, let us remark that the assumption we will make is stronger than just asking the group not to be simply laced: to define the following notions, the characteristic needs to be $p = 2$ for types B_n, C_n and F_4, and $p = 3$ in type G_2. Equivalently, the group G has Dynkin diagram having an edge of multiplicity p. From now on, let us call this the *edge hypothesis*. The following result is [CGP, Lemma 7.1.2].

Lemma 2.5.5. *Let G be simply connected satisfying the edge hypothesis. Then the vector subspace*

$$\mathfrak{n}_G := \langle \operatorname{Lie} \gamma^\vee(\mathbf{G}_m) \colon \gamma \in \Phi_< \rangle \oplus \mathfrak{g}_<$$

is a p-Lie ideal of $\operatorname{Lie} G$. *Moreover, every nonzero G-submodule of* $\operatorname{Lie} G$ *distinct from* $\operatorname{Lie} Z(G)$ *contains $\mathfrak{n}_G$.*

By the equivalence of categories between p-Lie subalgebras of $\operatorname{Lie} G$ and algebraic subgroups of G of height one (see Theorem 2.2.8), the p-Lie ideal $\mathfrak{n}_G$ lifts to a unique normal subgroup of G.

Notation 2.5.6. Let G be simply connected satisfying the edge hypothesis. The algebraic subgroup of height one having $\mathfrak{n}_G$ as Lie algebra is denoted as N_G.

The subgroup N_G is characterized by being the unique minimal nontrivial normal subgroup of G having trivial Frobenius; moreover, it is noncentral. For more details see [CGP, Definition 7.1.3]. Thus, we are led to consider the homomorphism

$$\pi_G \colon G \longrightarrow \overline{G} := G/N_G.$$

Let us remark that this is a noncentral isogeny with corresponding values $q_< = p$ and $q_> = 1$.

Definition 2.5.7. With the above notations, the homomorphism π_G is called the *very special isogeny* associated to the simple and simply connected algebraic group G.

The following step towards a better understanding of isogenies is the natural generalization of the above notion to the non simply connected case.

Definition 2.5.8. Let G be simple satisfying the edge hypothesis and let $\psi \colon \widetilde{G} \longrightarrow G$ be its simply connected cover. Let $N_{\widetilde{G}}$ be the kernel of the very special isogeny of $\widetilde{G}$ defined just above. We denote as:

- N_G its schematic image via the central isogeny ψ ;
- ${}_mN_{\widetilde{G}} := \ker(\pi_{\widetilde{G}^{(m)}} \circ F^m_{\widetilde{G}}) = (F^m_{\widetilde{G}})^{-1}(N_{\widetilde{G}^{(m)}})$, for any $m \geq 1$;
- ${}_mN_G$ the schematic image of ${}_mN_{\widetilde{G}}$ via the central isogeny ψ.

Let us remark that N_G is nontrivial, normal and has trivial Frobenius. Moreover, it is minimal with such properties: let $H \subset N_G$ be another such subgroup, then $\widetilde{H} := \psi^{-1}(H) \cap N_{\widetilde{G}}$ is nontrivial, normal and of height one, hence by definition contained in $N_{\widetilde{G}}$. This shows that $N_G = \psi(N_{\widetilde{G}}) \subset \psi(\widetilde{H}) = H$.
It is now natural to ask ourselves if such a subgroup is unique, or if we can give an example of it appearing in a natural context. This is shown in Lemma 2.5.15 and Example 2.5.16 below.

Up to this point in this section we have assumed that the Dynkin diagram of G has an edge of multiplicity p. What about the other cases not satisfying the edge hypothesis, in particular those which are not treated in [**Wen**]? Let us assume that either $p = 3$ and that the group G is simple of type B_n, C_n or F_4, or that $p = 2$ and the group G is simple of type G_2. Then an analogous construction to the subgroup N_G cannot be done for the following reason: nontrivial normal subgroups of height one correspond, under the equivalence of categories, to nonzero p-Lie ideals of $\operatorname{Lie} G$, which do not exist due to the following result (see [**Str**, 4.4]).

Lemma 2.5.9. *Let $p = 3$ and G be simple of type B_n, C_n for some $n \geq 2$, or F_4, or let $p = 2$ and G simple of type G_2. Then* $\operatorname{Lie} G$ *is simple as a p-Lie algebra.*

2.5.1.2. *Factorising isogenies.* Let us start by recalling the following result concerning the factorisation of the Frobenius morphism (see [**CGP**, Proposition 7.1.5]):

Proposition 2.5.10. *Let G be simple and simply connected satisfying the edge hypothesis; let π_G denote its very special isogeny. Then*

(a) There is a factorisation of the Frobenius morphism as

$$F_G\colon (G \xrightarrow{\quad \pi_G \quad} \overline{G} \xrightarrow{\quad \overline{\pi} \quad} G^{(1)})$$

which is the only nontrivial factorisation into isogenies with first step admitting no nontrivial factorisation into isogenies.

(b) The group $\overline{G}$ is simply connected, with root system $\overline{\Phi}$ isomorphic to the dual of the root system of G.

(c) The bijection between Φ and $\overline{\Phi}$ defined by π_G exchanges long and short roots: denoting it as $\alpha \leftrightarrow \overline{\alpha}$, if α is long then $\overline{\alpha}$ is short and vice-versa.

(d) In the factorisation of point (a), the map $\overline{\pi}$ is the very special isogeny of $\overline{G}$.

In particular, the restriction $(\pi_G)_{|U_\alpha}\colon U_\alpha \to U_{\overline{\alpha}}$ gives an isomorphism whenever α is long and a purely inseparable isogeny of degree p whenever α is short.

Lemma 2.5.11. *Assume $f\colon G \to G'$ is a noncentral isogeny with G simply connected and satisfying the edge hypothesis. If at least one value of $q(\alpha)$ is equal to 1, then necessarily $q_> = 1$.*

Proof. Let us start by proving the following: if all $q(\alpha)$s are equal to 1, then f is central. By definition of such integers, if $q_< = q_> = 1$ then $\ker f$ does not intersect the root subgroup U_γ for any root γ. Since $\ker f$ is normalized by the maximal torus T, this implies that $\ker f$ is itself contained in T, which means exactly that the isogeny is central.

By our hypothesis, we must hence have at least one value of q which is strictly greater than 1. Hence, it is enough to get a contradiction with the assumption

$$q_{>} \neq 1 \quad \text{and} \quad q_{<} = 1. \tag{2.5.3}$$

Let us assume that (2.5.3) holds and consider the subspace $\mathrm{Lie}(\ker f)$. The fact that $\ker f$ is a normal subgroup, together with the equivalence of categories of Theorem 2.2.8, means that $\mathrm{Lie}(\ker f)$ is a p-Lie ideal of the Lie algebra $\mathfrak{g}$, hence in particular it is a G-submodule of $\mathfrak{g}$ under the adjoint action. Moreover, it is stricly contained in $\mathfrak{g}$ because (2.5.3) means that the whole Frobenius kernel ${}_1G$ is not contained in $\ker f$. More precisely, (2.5.3) translates into

$$\mathfrak{g}_{>} \subset \mathrm{Lie}(\ker f) \quad \text{and} \quad \mathfrak{g}_{<} \cap \mathrm{Lie}(\ker f) = 0.$$

However, this gives a contradiction with Lemma 2.5.5, and we are done. □

Proposition 2.5.12. *Let G be a simple and simply connected algebraic group and let $f\colon G \to G'$ be an isogeny. Then there exists a unique factorisation of f as*

$$f\colon G \xrightarrow{\quad\pi\quad} \overline{G} \xrightarrow{\quad F^m_{\overline{G}}\quad} (\overline{G})^{(m)} \xrightarrow{\quad\rho\quad} G',$$

where ρ is a central isogeny and π is either the identity or - when G satisfies the edge hypothesis - the very special isogeny π_G.

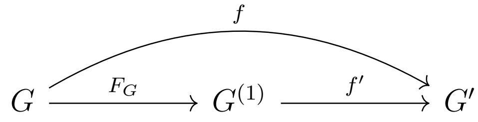
Remark 2.5.13. In particular, this allows us to speak of isogenies *with no central factor*, which are the ones we are interested in when classifying parabolic subgroups. Kernels of such isogenies are totally ordered by inclusion, as follows:

$$1 \subsetneq N_G \subsetneq {}_1G \subsetneq {}_1N_G \subsetneq \dots \subsetneq {}_mG \subsetneq {}_mN_G \subsetneq {}_{m+1}G \subsetneq \dots,$$

where ${}_mN_G$ (which we shall denote simply by ${}_mN$ when G is implicit) is the kernel of the composition of a very special isogeny and an m-th iterated Frobenius morphism.

Proof. (of Proposition 2.5.12).
Let us start by considering the bijection $\Phi \leftrightarrow \Phi'$ and the corresponding integers $q(\alpha)$ associated to the isogeny f, as recalled in (2.5.1).
Step 1: is the isogeny central? This is equivalent to asking whether all integers $q(\alpha)$ are equal to one. If this is the case, then we are done. Next, we hence assume that at least one value of q is nontrivial.
Step 2: does p divide $q(\alpha)$ for all roots α? If the group is simply laced this is always the case, since q is constant. If $p = 3$ and the group is of type B_n, C_n or F_4, or if $p = 2$ and the group is of type G_2, this is also always the case: indeed, there exists at least one $\gamma \in \Phi$ such that $q(\gamma) \neq 1$. Equivalently, the corresponding root space satisfies $\mathfrak{g}_\gamma \subset \mathfrak{h} := \mathrm{Lie}(\ker f)$. Since $\mathfrak{h}$ is a nontrivial p-Lie ideal of $\mathrm{Lie}\, G$, it must coincide with all of $\mathrm{Lie}\, G$ thanks to Lemma 2.5.9.
In general, if the answer is yes, then the root subspace $\mathfrak{g}_\alpha$ is contained in $\mathrm{Lie}(\ker f)$ for all roots. Since the latter is a Lie ideal of $\mathrm{Lie}\, G$, taking brackets implies that the copy of $\mathfrak{sl}_2$ associated to each root is also contained in $\mathrm{Lie}(\ker f)$, which thus coincides with $\mathrm{Lie}\, G$.

This means in particular that the Frobenius kernel of G is contained in the kernel of f, so we can factorise by the Frobenius morphism as follows



and go back to Step 1 replacing f by f'. Notice that this is possible, since the group $G^{(1)}$ is still simple and simply connected. Moreover, the new integers associated to the isogeny f' are exactly $q(\alpha)/p$, hence their values strictly decrease. After this step, we can hence assume that there are two distinct values $q_<$ and $q_>$ as defined in (2.5.2). In particular, let us remark that in this case G is not simply laced.

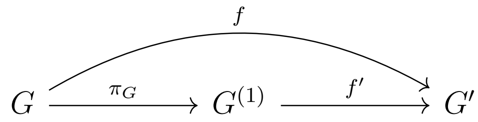
Step 3: all other cases are now settled using $\pi = \mathrm{id}_G$, so we may and do now assume that the Dynkin diagram of G has an edge of multiplicity p; moreover, by Lemma 2.5.11 $q_> = 1$ while $q_<$ is divisible by p. This last condition means that for any short root γ, the root subspaces $\mathfrak{g}_\gamma$ and $\mathfrak{g}_{-\gamma}$ are contained in $\mathrm{Lie}(\ker f)$. This implies that

$$(\mathfrak{sl}_2)_\gamma = [\mathfrak{g}_\gamma, \mathfrak{g}_{-\gamma}] \oplus \mathfrak{g}_\gamma \oplus \mathfrak{g}_{-\gamma} = \mathrm{Lie}(\gamma^\vee(\mathbf{G}_m)) \oplus \mathfrak{g}_\gamma \oplus \mathfrak{g}_{-\gamma} \subset \mathrm{Lie}(\ker f),$$

hence, by definition of the subgroup N_G in the simply connected case, we have

$$\langle \mathrm{Lie}(\gamma^\vee(\mathbf{G}_m)) \colon \gamma \in \Phi_< \rangle \bigoplus_{\gamma \in \Phi_<} \mathfrak{g}_\gamma =: \mathrm{Lie}\, N_G \subset \mathrm{Lie}(\ker f).$$

Since N_G is of height one, this implies that $N_G \subset \ker f$, so we can factorise by the very special isogeny as follows



and go back to Step 1. Notice that this is possible, since by Proposition 2.5.10, the group $\overline{G}$ is still simple and simply connected. Moreover, we know that the bijection $\Phi \leftrightarrow \overline{\Phi}$ exchanges long and short roots and that $(\pi_G)_{|U_\alpha}$ is an isomorphism for α long, while it is of degree p when α is short. By denoting as $q'(-)$ the integers associated to the new isogeny f', we then have

$$(q')_< = q'(\overline{\alpha}) = q(\alpha) = q_> = 1, \qquad (\alpha \text{ long})$$
$$(q')_> = q'(\overline{\alpha}) = q(\alpha)/p = q_</p, \qquad (\alpha \text{ short})$$

so the nontrivial integer strictly decreases after this step.

Following this procedure, one necessarily factorises a finite number of times leading finally to a central isogeny, which is the ρ given in the statement of the proposition. It remains to show that the Frobenius morphism and the very special isogeny - when it is defined - commute, in the following sense: if G is simple and simply connected, then

$$\pi_{G^{(1)}} \circ F_G = F_{\overline{G}} \circ \pi_G.$$

To prove this, let us apply the factorisation of the Frobenius morphism given in Proposition 2.5.10 twice to get

$$\pi_{G^{(1)}} \circ F_G = \pi_{G^{(1)}} \circ (\pi_{\overline{G}} \circ \pi_G) = (\pi_{G^{(1)}} \circ \pi_{\overline{G}}) \circ \pi_G = F_{\overline{G}} \circ \pi_G.$$

This means that we can commute π with the Frobenius and assume that it is the first morphism (or the middle one, which gives another unique factorisation) in the expression $f = \rho \circ F^m \circ \pi$. $\square$

Remark 2.5.14. The above Proposition allows us to associate to any isogeny $f\colon G \to G'$ between simple algebraic groups a diagram of the form

$$\begin{array}{ccc} G & \xrightarrow{\ f\ } & G' \\ {\scriptstyle\psi}\uparrow & & \uparrow{\scriptstyle\rho} \\ \widetilde{G} & \xrightarrow{\ F^m\circ\pi\ } & \widetilde{G'} \end{array}$$

where ψ is the simply connected cover of G and ρ is central. In particular, notice that the group

$$\overline{(\widetilde{G})}^{(m)},$$

which is the target of the morphism $F^m \circ \pi$, is simply connected (thanks to Proposition 2.5.10(b)) and ρ is central, thus this group is the simply connected cover of G'.

The first immediate consequence of this factorisation result is the uniqueness of the subgroup N_G.

Lemma 2.5.15. *Let G be simple satisfying the edge hypothesis and $H \subset G$ a normal, noncentral subgroup of height one. Then H contains the subgroup N_G. In particular, such a subgroup N_G is unique.*

Proof. The conclusion clearly holds when H equals the Frobenius kernel of G, hence we can assume that $H \neq {}_1G$. To prove that $N_G \subset H$ it is enough to show that $f(N_G)$ is trivial, where f is the isogeny $G \to G/H$. Consider the associated diagram given in Remark 2.5.14:

$$\begin{array}{ccc} G & \xrightarrow{\ f\ } & G/H \\ {\scriptstyle\psi}\uparrow & & \uparrow{\scriptstyle\rho} \\ \widetilde{G} & \xrightarrow{\ F^m\circ\pi\ } & \widetilde{G/H} \end{array}$$

where π is either the identity or the very special isogeny of G. We want to show that the bottom arrow is necessarily the very special isogeny $\pi_{\widetilde{G}}$. First, the subgroup H is noncentral hence if $m = 0$ then $\pi = \pi_{\widetilde{G}}$, otherwise the bottom row would be the identity and f would be central. Moreover, $H \subsetneq {}_1G = \ker(F\colon G \to G^{(1)})$ hence the factorisation of the isogeny $f \circ \psi$ in the above diagram must satisfy $m = 0$. Thus, we can conclude that $f \circ \psi = \rho \circ \pi_{\widetilde{G}}$ and

$$f(N_G) = f(\psi(N_{\widetilde{G}})) = \rho(\pi_{\widetilde{G}}(N_{\widetilde{G}})) = 1$$

as wanted. $\square$

Example 2.5.16. Let us assume $p = 2$ and consider the group $G = \mathrm{SO}_{2n+1} = \mathrm{SO}(k^{2n+1})$ in type B_n with $n \geq 2$, defined as being relative to the quadratic form

$$Q(x) = x_n^2 + \sum_{i=0}^{n-1} x_i x_{2n-i}$$

and $G' = \mathrm{Sp}_{2n} = \mathrm{Sp}(k^{2n})$ relative to the skew form

$$b(y, y') = \sum_{i=1}^{n} y_i y'_{2n+1-i} - y_{2n+1-i} y'_i.$$

Since G fixes the middle vector of the canonical basis e_n, it acts on $k^{2n} = k^{2n+1}/ke_n$ and this gives an isogeny

$$\varphi\colon G = \mathrm{SO}_{2n+1} \longrightarrow \mathrm{Sp}_{2n} = G',$$

of degree 2^{2n}. Since the target of the isogeny is already simply connected, the diagram of Remark 2.5.14 is as follows:

$$\begin{array}{ccc} \mathrm{SO}_{2n+1} & \xrightarrow{\ \varphi\ } & \mathrm{Sp}_{2n} \\ {\scriptstyle\psi}\uparrow & \nearrow_{F^m\circ\pi} & \\ \mathrm{Spin}_{2n+1} & & \end{array}$$

In particular, since ψ is central - identifying the root systems of a group and of its simply connected cover - the integers $q(-)$ associated to the isogeny φ must be the same as those associated to the composition $F^m \circ \pi$. In particular, this implies $m = 0$; hence the subgroup

$$N_{\mathrm{SO}_{2n+1}} = \ker \varphi = \psi(\ker \pi) = \psi(N_{\mathrm{Spin}_{2n+1}}),$$

which appears in this natural construction, coincides with the one just defined above. In particular, in this case

$$\operatorname{Lie} N_{\mathrm{Spin}_{2n+1}} = \operatorname{Lie}(\varepsilon_n^\vee(\mathbf{G}_m)) \bigoplus_{\gamma \in \Phi_<} \mathfrak{g}_\gamma = \operatorname{Lie}(\varepsilon_n^\vee(\mathbf{G}_m)) \bigoplus_{1 \le i \le n} (\mathfrak{g}_{-\varepsilon_i} \oplus \mathfrak{g}_{\varepsilon_i}.)$$

To conclude this example, let us determine explicitly the subgroup $N_{\mathrm{SO}_{2n+1}} = \ker \varphi$ and its Lie algebra, which is needed later on. A matrix in $\ker \varphi$ is of the form

$$A = \begin{pmatrix} & a_0 & \\ 1_n & \vdots & 0_n \\ & a_{n-1} & \\ b_0 \dots b_{n-1} & a_n & b_{n+1} \dots b_{2n} \\ & a_{n+1} & \\ 0_n & \vdots & 1_n \\ & a_{2n} & \end{pmatrix}$$

and the condition for A to be in SO_{2n+1} gives

$$\begin{aligned} Q(Ax) &= a_n^2 x_n + \sum_{j \neq n} b_j^2 x_j^2 + \sum_{i=0}^{n-1} (x_i x_{2n-i} + a_{2n-i} x_i x_n + a_i x_{2n-i} x_n) \\ &= Q(x) = x_n^2 + \sum_{i=0}^{n-1} x_i x_{2n-i}, \end{aligned}$$

which is equivalent to $a_i = 0$ for all $i \neq n$, $a_n^2 = 1$ and $b_i^2 = 0$ for all i. Moreover, under these conditions $\det A = a_n = 1$, thus we have

$$N_{\mathrm{SO}_{2n+1}} = \ker\varphi = \left\{ \begin{pmatrix} & 1 & 0 & 0 & \\ b_0 \dots b_{n-1} & & 1 & & b_{n+1} \dots b_{2n} \\ & 0 & 0 & 1 & \end{pmatrix} \in \mathrm{GL}_{2n+1} \colon b_i \in \alpha_p \right\} \simeq \alpha_p^{2n}.$$

Finally, using the equalities in Remark 3.1.11 concerning short roots, we can conclude that $\mathrm{Lie}\, N_{\mathrm{SO}_{2n+1}} = \mathfrak{g}_<$.

CHAPTER 3

Parabolic subgroups having maximal reduced part

Abstract. We extend to characteristic 2 and 3 the classification of projective homogeneous varieties of Picard group **Z**, corresponding to parabolic subgroups with maximal reduced subgroup. In all types, except for G_2 in characteristic 2, the latter are all obtained as product of a maximal reduced parabolic with the kernel of a purely inseparable isogeny. For the G_2 case, we exhibit an explicit counterexample and show it is the only one, thus completing the classification.

3.1. Classification in all types but G_2

Let us recall that we are working with a semisimple algebraic group $G \supset B \supset T$ over an algebraically closed field k of characteristic $p > 0$, together with a fixed Borel subgroup B and a maximal torus T contained in it. Our aim is to prove that all projective homogeneous varieties under a G-action having Picard group of rank one are isomorphic (as varieties) to homogeneous spaces having reduced stabilizers, in every type except G_2 when the characteristic is $p = 2$.

Let us remark that, since the Picard rank of $X = G/P$ is equal to the number of simple roots of G not contained in the root system of a Levi subgroup of P, such spaces are realized as quotients G/P such that the reduced subgroup of the stabilizer P is maximal. For a full justification of this assertion, see Section 5.1.
The main result is the following :

Theorem 3.1.1. *Let X be a projective algebraic variety over an algebraically closed field of characteristic $p > 0$, homogeneous under a faithful action of a smooth connected algebraic group H and having Picard group isomorphic to* **Z**.
Then there is a simple adjoint algebraic group G and a reduced *maximal parabolic subgroup $P \subset G$ such that $X = G/P$, unless $p = 2$ and H is of type G_2.*

The purpose of this Section is to prove the above Theorem: the idea is to do it explicitly case by case, since there seems to be no easy general geometric argument, as the case of type G_2 in characteristic two confirms. We proceed as follows: in Section 3.1.1 we perform elementary reductions to the case where $X = G/P$ with G simple and the characteristic is 2 or 3, and we recall some notation and results used in the proof. In Section 3.1.2 we illustrate the strategy of the proof in the simplest case of type A_{n-1}. In Sections 3.1.3 to 3.1.5 we implement the argument in types B_n, C_n and F_4. The case of G_2, for which the above Theorem fails in characteristic 2, is then studied separately in Section 3.2.

3.1.1. Reductions and notation. Let us place ourselves under the hypothesis of Theorem 3.1.1 and denote as H_{aff} the largest smooth connected affine normal subgroup of H. By [**BSU**, Theorem 4.1.1], there is a canonical isomorphism $X \simeq A \times Y$, where A is an abelian variety and Y is a projective homogeneous variety under a faithful H_{aff} action. Moreover, H_{aff} is semisimple and of adjoint type. Under our assumptions, the abelian variety must be a point because otherwise the Picard group of X would not be discrete. More precisely, the hypothesis $\operatorname{Pic} X = \mathbf{Z}$ implies that we can assume H to be simple, thanks to the combinatorial description of the Białynicki-Birula decomposition of homogeneous spaces (of which a detailed statement is given in Theorem 5.1.9). After such reductions, it is thus enough to prove the following statement.

THEOREM 3.1.2. *Let G be a simple adjoint group, not of type G_2 when the characteristic is 2, and P a parabolic subgroup such that P_{red} is maximal. If G acts faithfully on $X = G/P$, then P is a* reduced *parabolic subgroup.*

Let us keep notations from Section 2.4. In particular, P^α denotes the maximal reduced parabolic subgroup not containing $U_{-\alpha}$, for α a simple root, while ${}_mG$ denotes the kernel of the m-th iterated relative Frobenius homomorphism of G. Let us also recall for reference the statement of [**Wen**, Theorem 14].

PROOF. (of Theorem 3.1.1 assuming Theorem 3.1.2) Let us consider $G_0 = \underline{\mathrm{Aut}}^0_X$ to be the neutral component of the reduced automorphism group of X, as in Remark 2.3.1. The group G_0 is a simple group of adjoint type, acting faithfully on X. Then by the assumption on the Picard group of X we have $X = G_0/P_0$ for some parabolic subgroup P_0 of G_0 such that $P_{0,\mathrm{red}} = P^\alpha$ for some simple root α of G_0. Then one can apply Theorem 3.1.2 to deduce that P_0 actually coincides with its reduced part, and we are done. □

THEOREM 3.1.3. *There is an injective map*

$$\mathrm{Hom}_{Set}(\Delta, \mathbf{N} \cup \{\infty\}) \longrightarrow \{\textit{parabolic subgroups } G \supset P \supset B\}$$
$$\varphi \longmapsto \bigcap_{\alpha\in\Delta:\ \varphi(\alpha)\neq\infty} {}_{\varphi(\alpha)}GP^\alpha.$$

Moreover, if $p \geq 5$ or the Dynkin diagram of G is simply laced, this map is also surjective.

Remark 3.1.4. Let us start by taking a projective variety X which is homogeneous under the action of a simple group H. By replacing such a group with the image G of the morphism $H \to \underline{\mathrm{Aut}}_X$ (see Remark 2.3.1 concerning the notation on automorphism groups) we may assume that the action is faithful. In particular, this means that there is no normal algebraic subgroup of G contained in P. However, we need to be careful in the case-by-case proof because this additional assumption - which is not restrictive on the varieties considered - forces the group G to be of adjoint type.

Let us place ourselves in the setting of Theorem 3.1.2 and sketch the strategy of the proof: let P be a nonreduced parabolic subgroup such that

$$P_{\mathrm{red}} = P^\alpha$$

for some simple positive root $\alpha \in \Delta$; consider $P^\alpha \subsetneq P \subset G$, inducing the corresponding inclusions on Lie algebras:

$$\operatorname{Lie} P^\alpha \subsetneq \operatorname{Lie} P \subset \operatorname{Lie} G.$$

Since we do not have any information a priori on P, we study the quotient

$$V_\alpha := \operatorname{Lie} G / \operatorname{Lie} P^\alpha,$$

considered as a L^α-module under the representation given by the adjoint action, where L^α denotes the Levi subgroup defined as the intersection $P^\alpha \cap (P^\alpha)^-$ with the corresponding opposite parabolic subgroup.

Let us recall some notation and state a Lemma on structure constants which will be repeatedly used in what follows:

- the decomposition of the Lie algebra in weight spaces under the T-action is
$$\mathfrak{g} = \operatorname{Lie} G = \operatorname{Lie} T \oplus \left(\bigoplus_{\gamma \in \Phi} \mathfrak{g}_\gamma \right),$$
- when G is simply connected, a Chevalley basis of $\operatorname{Lie} G$ is denoted as
$$\{X_\gamma, H_\alpha\}_{\gamma \in \Phi, \alpha \in \Delta}.$$

In particular, $\mathfrak{g}_\gamma = \operatorname{Lie} U_\gamma = kX_\gamma$ and $X_\gamma = \mathrm{d}u_\gamma(1)$, where u_γ is the root homomorphism $\mathbf{G}_a \xrightarrow{\sim} U_\gamma$. Whenever the Dynkin diagram of G is not simply laced,

- $\Phi_< \subset \Phi$ and $\Phi_> \subset \Phi$ denote respectively the subsets of short and long roots, whenever a multiple edge appears in the Dynkin diagram,
- when G is simply connected and satisfies the edge hypothesis, N_G denotes the finite group scheme of height one whose Lie algebra is given by
$$\operatorname{Lie} N_G = \langle \operatorname{Lie}(\gamma^\vee(\mathbf{G}_m)) \colon \gamma \in \Phi_< \rangle \oplus \left(\bigoplus_{\gamma \in \Phi_<} \mathfrak{g}_\gamma \right),$$
as seen in Section 2.5.
- when G is not simply connected and satisfies the edge hypothesis, N_G denotes the schematic image of $N_{\widetilde{G}}$ via the universal covering map, where $\widetilde{G}$ is the simply connected cover of G - see again Definition 2.5.8.

Let us state the following Lemma - see [**Hum**, Chapter VII, 25.2] - which allows us to calculate all structure constants with respect to a Chevalley basis of the Lie algebra $\operatorname{Lie} G$, where G is simple and simply connected.

Lemma 3.1.5 (Chevalley). *Let $\{X_\gamma \colon \gamma \in \Phi,\ H_\alpha \colon \alpha \in \Delta\}$ be a Chevalley basis for $\operatorname{Lie} G$, where G is simple and simply connected. Then the resulting structure constants satisfy*

(a) $[H_\alpha, H_\beta] = 0$ *for all* $\alpha, \beta \in \Delta$ *;*
(b) $[H_\alpha, X_\gamma] = \langle \alpha, \gamma \rangle X_\gamma$ *for all* $\alpha \in \Delta$, $\gamma \in \Phi$ *;*
(c) $[X_{-\gamma}, X_\gamma]$ *is a linear combination with integer coefficients of the* H_α*'s ;*
(d) $[X_\gamma, X_\delta] = \pm(r+1)X_{\gamma+\delta}$ *for all* $\delta \neq \pm\gamma$ *roots such that the* δ*-string through* γ *goes from* $\gamma - r\delta$ *to* $\gamma + q\delta$ *with* $q \geq 1$*, i.e. such that* $\gamma + \delta$ *is still a root ;*
(e) $[X_\gamma, X_\delta] = 0$ *for all roots* $\delta \neq \pm\gamma$ *such that* $\gamma + \delta$ *is not a root.*

In particular, the Chevalley relation we use the most frequently is (d): it is important to recall that structure constants appearing in such equations are among $\pm 1, \pm 2, \pm 3, \pm 4$, which indicates why problems arise in characteristic 2 and 3.

The main line of argument to prove Theorem 3.1.2 is the following: we start by considering $X = G/P$ with G adjoint acting faithfully and P nonreduced. Then with some computations on Lie algebras, we show that - when it is defined - $N_G \subset P$, while otherwise ${}_1G \subset P$. In both cases this gives a normal algebraic subgroup of G contained in the stabilizer P, which cannot exist due to Remark 3.1.4.

3.1.2. Type A_{n-1}. We start with a case whose classification is already covered by [Wen] - without needing any assumption on the characteristic of the base field - but which is useful in order to explain the approach used in the other cases below.

Let us consider the reductive group $G = \mathrm{GL}_n$ in type A_{n-1}, its maximal torus T given by diagonal matrices of the form

$$t = \mathrm{diag}(t_1, \dots, t_n) \in \mathrm{GL}_n$$

and the Borel subgroup B of upper triangular matrices. Let us denote as $\varepsilon_i \in X(T)$ the character sending $t \mapsto t_i$, for $i = 1, \dots, n$. Then the root system $\Phi = \Phi(G, T)$ is given by

$$\Phi^+ = \{\varepsilon_i - \varepsilon_j,\ 1 \leq i < j \leq n\},$$

with basis Δ consisting of the following roots :

$$\alpha_1 = \varepsilon_1 - \varepsilon_2,\ \dots,\ \alpha_{n-1} = \varepsilon_{n-1} - \varepsilon_n.$$

Finally, assume given a nonreduced parabolic subgroup P such that $P_{\mathrm{red}} = P_m$, where $P_m := P^{\alpha_m}$ denotes the maximal reduced parabolic subgroup associated to the simple positive root α_m for a fixed $1 \leq m < n$. Thus, the Levi subgroup L_m of this reduced parabolic subgroup is a product of a reductive group of type A_{m-1} and one of type A_{n-m-1}:

$$L_m = \left\{ \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \right\} \simeq \mathrm{GL}_m \times \mathrm{GL}_{n-m},$$

and the two factors have as basis of simple roots $\{\alpha_1, \dots, \alpha_{m-1}\}$ and $\{\alpha_{m+1}, \dots, \alpha_{n-1}\}$ respectively.

Now, let us consider the vector space $V_m = \mathrm{Lie}\, G / \mathrm{Lie}\, P_m$. Since

$$\{\gamma \in \Phi^+ \colon \alpha_m \in \mathrm{Supp}(\gamma)\} = \{\varepsilon_i - \varepsilon_j,\ i \leq m < j\},$$

the root spaces in V_m are of the form $\mathfrak{g}_{-\varepsilon_i + \varepsilon_j} = k E_{ji}$, for $i \leq m < j$, where E_{ji} denotes the square matrix of order n having all zero entries except the (j, i)-th entry which is equal to 1. Concretely, V_m is isomorphic as L_m-module to the space of all matrices M of size $(n - m) \times m$. The action of L_m on V_m is given by

$$(A, B) \cdot M = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} 0 & 0 \\ M & 0 \end{pmatrix} \begin{pmatrix} A^{-1} & 0 \\ 0 & B^{-1} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ BMA^{-1} & 0 \end{pmatrix} = BMA^{-1},$$

for all $A \in \mathrm{GL}_m$, $B \in \mathrm{GL}_{n-m}$. This just corresponds to the natural action of $\mathrm{GL}_m \times \mathrm{GL}_{n-m}$ on $\mathrm{Hom}_k(k^m, k^{n-m})$. In particular, V_m is an irreducible L_m-module.

Proof. (**of Theorem 3.1.2 in type A_{n-1}**)
Let $G = \mathrm{PGL}_n$ and let $X = G/P$ such that G acts faithfully on X, P is non-reduced and has as reduced part a maximal smooth parabolic subgroup. By pulling back both G and P to the reductive group GL_n, we get $X = \mathrm{GL}_n / Q$, where $Q_{\mathrm{red}} = P^{\alpha_m}$ for some m. Since $\mathrm{Lie}\, Q / \mathrm{Lie}\, P_m$ is an L_m-submodule of V_m,

$$\mathrm{Lie}\, Q = \mathrm{Lie}\, \mathrm{GL}_n$$

hence by Theorem 2.2.8 the Frobenius kernel ${}_1\mathrm{GL}_n$ is contained in Q. By considering the images into the quotient PGL_n, we get that P contains a nontrivial normal subgroup of G of height one. Thus, we get a contradiction by Remark 3.1.4. Hence, the parabolic P needs to be reduced. □

In other words, under the hypothesis of maximality of the reduced subgroup, we find that there are no new varieties other than those of the known classification. In the following subsections we will treat the other cases - not included in Wenzel's article - where two different root lengths are involved.

Remark 3.1.6. What does this case correspond to, geometrically, on the level of varieties? We know by [**Wen**] that $P_{\mathrm{red}} = P^{\alpha_m}$ implies $P = {}_rGP^{\alpha_m}$ for some $r \geq 0$, hence

$$X = G/{}_rGP^{\alpha_m} \simeq G^{(r)}/(P^{\alpha_m})^{(r)} \simeq G/P^{\alpha_m} = \mathrm{Grass}_{m,n}$$

is isomorphic to the Grassmannian of m-th dimensional vector subspaces in k^n, equipped with the natural $G = \mathrm{GL}_n$-action, twisted by the r-th iterated Frobenius morphism. In particular, the assumption of faithfulness of the action implies $r = 0$.

3.1.3. Type C_n. Let us consider the group $\widetilde{G} = \mathrm{Sp}_{2n}$ in type C_n, with $n \geq 2$ in characteristic $p = 2$ or 3. Defining $\widetilde{G}$ as relative to the skew form

$$b(x,y) = \sum_{i=1}^{n} x_i y_{2n+1-i} - x_{2n+1-i} y_i$$

on k^{2n}, one has

$$\widetilde{G} = \left\{ X \in \mathrm{GL}_{2n} \colon {}^tX \begin{pmatrix} 0 & \Omega_n \\ -\Omega_n & 0 \end{pmatrix} X = \begin{pmatrix} 0 & \Omega_n \\ -\Omega_n & 0 \end{pmatrix} \right\}, \quad \text{where } \Omega_n = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \iddots & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Deriving this condition gives as Lie algebra

$$\begin{aligned} \mathrm{Lie}\, \widetilde{G} &= \left\{ M \in \mathfrak{gl}_{2n} \colon {}^tM \begin{pmatrix} 0 & \Omega_n \\ -\Omega_n & 0 \end{pmatrix} + \begin{pmatrix} 0 & \Omega_n \\ -\Omega_n & 0 \end{pmatrix} M = 0 \right\} \\ &= \left\{ \begin{pmatrix} A & B \\ C & -A^\sharp \end{pmatrix} \in \mathfrak{gl}_{2n} \colon B = B^\sharp \text{ and } C = C^\sharp \right\}, \end{aligned}$$

where for any square matrix X of order n we denote as $X^\sharp$ the matrix $\Omega_n {}^tX \Omega_n$, i.e.

$$(X^\sharp)_{i,j} = X_{n+1-j,n+1-i}. \tag{3.1.1}$$

Remark 3.1.7. Next, let us consider as maximal torus T the one given by diagonal matrices of the form

$$t = \operatorname{diag}(t_1, \ldots, t_n, t_n^{-1}, \ldots, t_1^{-1}) \in \mathrm{GL}_{2n}$$

and denote as $\varepsilon_i \in X^*(T)$ the character sending $t \mapsto t_i$, for $i = 1, \ldots, n$. A direct computation gives the following root spaces in $\operatorname{Lie} \widetilde{G}$:

$$\begin{aligned}
\mathfrak{g}_{2\varepsilon_i} &= k\begin{pmatrix} 0 & E_{i,n+1-i} \\ 0 & 0 \end{pmatrix} = k\begin{pmatrix} 0 & E_{ii}\Omega_n \\ 0 & 0 \end{pmatrix} \\
\mathfrak{g}_{-2\varepsilon_i} &= k\begin{pmatrix} 0 & 0 \\ E_{n+1-i,i} & 0 \end{pmatrix} = k\begin{pmatrix} 0 & 0 \\ \Omega_n E_{ii} & 0 \end{pmatrix}, && 1 \leq i \leq n, \\
\mathfrak{g}_{\varepsilon_i+\varepsilon_j} &= k\begin{pmatrix} 0 & E_{i,n+1-j} + E_{j,n+1-i} \\ 0 & 0 \end{pmatrix} = k\begin{pmatrix} 0 & (E_{ij}+E_{ji})\Omega_n \\ 0 & 0 \end{pmatrix}, \\
\mathfrak{g}_{-\varepsilon_i-\varepsilon_j} &= k\begin{pmatrix} 0 & 0 \\ E_{n+1-i,j} + E_{n+1-j,i} & 0 \end{pmatrix} = k\begin{pmatrix} 0 & 0 \\ \Omega_n(E_{ij}+E_{ji}) & 0 \end{pmatrix}, && i < j, \\
\mathfrak{g}_{\varepsilon_i-\varepsilon_j} &= k\begin{pmatrix} E_{ij} & 0 \\ 0 & -E_{n+1-j,n+1-i} \end{pmatrix} = k\begin{pmatrix} E_{ij} & 0 \\ 0 & -E_{ij}^\sharp \end{pmatrix}, \\
\mathfrak{g}_{-\varepsilon_i+\varepsilon_j} &= k\begin{pmatrix} E_{ji} & 0 \\ 0 & -E_{n+1-i,n+1-j} \end{pmatrix} k\begin{pmatrix} E_{ji} & 0 \\ 0 & -E_{ji}^\sharp \end{pmatrix}, && i < j,
\end{aligned}$$

where E_{ij} denotes the square matrix of order n with zero entries except for the (i,j)-th which is equal to one.

The root system $\Phi = \Phi(\widetilde{G}, T)$ is thus

$$\Phi^+ = \{\varepsilon_i - \varepsilon_j,\ \varepsilon_i + \varepsilon_j,\ 1 \leq i < j \leq n\} \cup \{2\varepsilon_i,\ 1 \leq i \leq n\},$$

having chosen as Borel subgroup the one given by all upper triangular matrices in $\widetilde{G} \subset \mathrm{GL}_{2n}$. The corresponding basis Δ consists of the following roots :

$$\alpha_1 = \varepsilon_1 - \varepsilon_2,\ \ldots,\ \alpha_{n-1} = \varepsilon_{n-1} - \varepsilon_n,\ \alpha_n = 2\varepsilon_n. \tag{3.1.2}$$

3.1.3.1. *Reduced parabolic P_n.* Still considering the group $\widetilde{G} = \mathrm{Sp}_{2n}$, denote as P_n the maximal reduced parabolic subgroup associated to the long simple positive root α_n: in a more intrinsic way, this subgroup is the stabilizer of an isotropic vector subspace $W \subset V$ of dimension n, where $\widetilde{G} = \mathrm{Sp}(V)$. In particular, W is the span of $e_1, \ldots, e_n$, where $(e_i)_{i=1}^{2n}$ denotes the standard basis of k^{2n}. Moreover, let us denote as P_n^- the opposite parabolic subgroup and as L_n their common Levi subgroup, so that

$$\begin{aligned}
P_n &= \operatorname{Stab}(W \subset V) \\
P_n^- &= \operatorname{Stab}(W^* \subset V) \\
L_n &= P_n \cap P_n^- = \mathrm{GL}(W) \simeq \mathrm{GL}_n,
\end{aligned}$$

where $W \oplus W^* = V$. Let us also remark that L has root system Ψ given by

$$\Psi^+ = \{\varepsilon_i - \varepsilon_j,\ 1 \leq i < j \leq n\},$$

corresponding to a reductive group of type A_{n-1} having as basis $\alpha_1, \ldots, \alpha_{n-1}$. This can be visualized in the following block decomposition :

$$L_n = \left\{ \begin{pmatrix} A & 0 \\ 0 & -(A^{-1})^\sharp \end{pmatrix} : A \in \mathrm{GL}(W) \simeq \mathrm{GL}_n \right\} \subset \widetilde{G}.$$

First, the Lie algebra of P_n is

$$\mathrm{Lie}\, P_n = \mathrm{Lie}\, B \oplus \left(\bigoplus_{i<j} \mathfrak{g}_{-\varepsilon_i+\varepsilon_j} \right) = \bigoplus_{i<j} (\mathfrak{g}_{\varepsilon_i-\varepsilon_j} \oplus \mathfrak{g}_{-\varepsilon_i+\varepsilon_j}) \oplus \left(\bigoplus_{i<j} \mathfrak{g}_{\varepsilon_i+\varepsilon_j} \right) \oplus \left(\bigoplus_{i} \mathfrak{g}_{2\varepsilon_i} \right).$$

For our purposes it is useful to study the L_n-action on the vector space

$$V_n := \mathrm{Lie}\, \widetilde{G} / \mathrm{Lie}\, P_n = \left(\bigoplus_{i<j} \mathfrak{g}_{-\varepsilon_i-\varepsilon_j} \right) \oplus \left(\bigoplus_{i} \mathfrak{g}_{-2\varepsilon_i} \right).$$

Lemma 3.1.8. *The L_n-module V_n is isomorphic to the dual of the standard representation of GL_n on $\mathrm{Sym}^2(k^n)$.*

Proof. Indeed, the root spaces we are interested in have been computed in Remark 3.1.7. Those equalities imply that a matrix in V_n is of the form

$$\begin{pmatrix} 0 & 0 \\ \Omega_n X & 0 \end{pmatrix}, \quad \text{with } X \in \mathrm{Sym}^2(k^n),$$

thus the dual action of $A \in \mathrm{GL}_n \simeq L_n$ can be computed as follows:

$${}^tA^{-1} \cdot X \simeq \begin{pmatrix} {}^tA^{-1} & 0 \\ 0 & -({}^tA)^\sharp \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \Omega_n X & 0 \end{pmatrix} \begin{pmatrix} {}^tA & 0 \\ 0 & -({}^tA^{-1})^\sharp \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -\Omega_n A X {}^tA & 0 \end{pmatrix} \simeq A X\, {}^tA.$$

This gives the desired isomorphism between the two GL_n-modules.
Let us remark that if we are working over a field of characteristic $p = 2$, the L_n-module V_n contains a simple L_n-submodule, namely

$$\left\{ \begin{pmatrix} & 0 & & 0 \\ & & c_1 & \\ & \iddots & & \\ c_n & & & 0 \end{pmatrix}, c_i \in k \right\} = \bigoplus_{i=1}^{n} \mathfrak{g}_{-2\varepsilon_i},$$

which is isomorphic to the dual of the standard representation of GL_n on k^n, twisted once by the Frobenius morphism. □

Proposition 3.1.9. *Assume given a nonreduced parabolic subgroup P such that $P_{red} = P_n$. Then $\mathrm{Lie}\, P = \mathrm{Lie}\, \widetilde{G}$ or $\mathrm{Lie}\, P = \mathrm{Lie}\, P_n + \mathfrak{g}_<$. If $p = 3$, then necessarily $\mathrm{Lie}\, P = \mathrm{Lie}\, \widetilde{G}$.*

Proof. Let us assume that $p = 2$ and consider the nonzero vector space $\mathrm{Lie}\, P / \mathrm{Lie}\, P_n$, which is an L_n-submodule of V_n. The latter being isomorphic to $\mathrm{Sym}^2(k^n)^*$ by Lemma 3.1.8, we have that

(a) either $\mathrm{Lie}\, P / \mathrm{Lie}\, P_n$ contains all of the weight spaces $\mathfrak{g}_{-2\varepsilon_i}$ associated to long negative roots,
(b) or it does not contain any of them.

Let us start by (a) and assume $\mathfrak{g}_{-2\varepsilon_i} \subset \operatorname{Lie} P$ for all i. In order to prove that $\operatorname{Lie} P = \operatorname{Lie} \widetilde{G}$, it is enough to show that for any $i < j$, the Chevalley vector $X_{-\varepsilon_i-\varepsilon_j}$ also belongs to $\operatorname{Lie} P$. For this, let us consider roots

$$\gamma = \varepsilon_i - \varepsilon_j, \quad \text{satisfying } X_\gamma \in \operatorname{Lie} L_n \subset \operatorname{Lie} P,$$
$$\delta = -2\varepsilon_i, \quad \text{satisfying } X_\delta \in \operatorname{Lie} P \text{ by our last assumption.}$$

Thus, $\gamma + \delta = -\varepsilon_i - \varepsilon_j$ is still a root while $\delta - \gamma = -3\varepsilon_i - \varepsilon_j$ is not: applying Lemma 3.1.5 gives

$$[X_{\varepsilon_i-\varepsilon_j}, X_{-2\varepsilon_i}] = \pm X_{-\varepsilon_i-\varepsilon_j} \in \operatorname{Lie} P$$

as wanted.

Let us place ourselves in the hypothesis of (b) and assume that no root subspace associated to a negative long root is in $\operatorname{Lie} P$. Since by assumption P is nonreduced, $\operatorname{Lie} P_n \subsetneq \operatorname{Lie} P$ so there must be at least one short root of the form $-\varepsilon_i - \varepsilon_j$ satisfying $X_{-\varepsilon_i-\varepsilon_j} \in \operatorname{Lie} P$. We will now prove that this implies all short roots $-\varepsilon_l - \varepsilon_m$ for $l < m$ belong to $\operatorname{Lie} P$, hence showing $\operatorname{Lie} P = \operatorname{Lie} P_n + \mathfrak{g}_<$.
First, assume $l \neq i, j$ and consider roots

$$\gamma = -\varepsilon_i - \varepsilon_j, \quad \text{satisfying } X_\gamma \in \operatorname{Lie} P \text{ by assumption,}$$
$$\delta = -\varepsilon_l + \varepsilon_i, \quad \text{satisfying } X_\delta \in \operatorname{Lie} L_n \subset \operatorname{Lie} P.$$

In this case, $\gamma + \delta = -\varepsilon_l - \varepsilon_j$ is still a root while $\delta - \gamma = -\varepsilon_l + 2\varepsilon_i + \varepsilon_j$ is not: applying Lemma 3.1.5 gives

$$[X_{-\varepsilon_i-\varepsilon_j}, X_{-\varepsilon_l+\varepsilon_i}] = \pm X_{-\varepsilon_l-\varepsilon_j} \in \operatorname{Lie} P.$$

Now, let us fix any $l < m$ satisfying $l, m \neq j$ and consider roots

$$\gamma = \varepsilon_j - \varepsilon_m, \quad \text{satisfying } X_\gamma \in \operatorname{Lie} L_n \subset \operatorname{Lie} P,$$
$$\delta = -\varepsilon_l - \varepsilon_j, \quad \text{satisfying } X_\delta \in \operatorname{Lie} P \text{ by the last step.}$$

Thus, $\gamma + \delta = -\varepsilon_l - \varepsilon_m$ is still a root while $\delta - \gamma = -\varepsilon_l - 2\varepsilon_j + \varepsilon_m$ is not: applying Lemma 3.1.5 gives

$$[X_{\varepsilon_j-\varepsilon_m}, X_{-\varepsilon_l-\varepsilon_j}] = \pm X_{-\varepsilon_l-\varepsilon_m} \in \operatorname{Lie} P.$$

If we are working over a field of characteristic $p = 3$, the representation of GL_n acting on $\mathrm{Sym}^2(k^n)$ is already an irreducible one. Let us justify this claim in a representation theoretic setting: for a finite dimensional vector space V', $\mathrm{Sym}^2(V')$ is the so-called *standard* representation of $\mathrm{GL}(V')$, defined as $H^0(2\varpi_1)$, where ϖ_1 is the first fundamental weight (see [**Jan**, II.2.16]). In particular, it is irreducible (i.e. it coincides with the simple $\mathrm{GL}(V')$-module associated to $2\varpi_1$) in any characteristic but $p = 2$; this means that in characteristic 3, V_n is an irreducible L_n-module. Hence the nonzero submodule $\operatorname{Lie} P / \operatorname{Lie} P_n$ must coincide with all of V_n. Equivalently, $\operatorname{Lie} P = \operatorname{Lie} \widetilde{G}$ as wanted. □

Proof. (**of Theorem 3.1.2 in type C_n when $P_{\mathrm{red}} = P_n$**)
Let G be simple adjoint of type C_n and $X = G/P$ with a faithful G-action such that $P_{\mathrm{red}} = P^{\alpha_n}$ and P is nonreduced. Define $\widetilde{P} \subset \widetilde{G} = \mathrm{Sp}_{2n}$ as being the preimage of P in

the simply connected cover: it is a nonreduced parabolic subgroup satisfying $\widetilde{P}_{\mathrm{red}} = P_n$. When $p = 2$, the above Proposition implies that

$$\langle\, \mathrm{Lie}(\gamma^\vee(\mathbf{G}_m))\colon \gamma \in \Phi_< \,\rangle \oplus \mathfrak{g}_< = \mathrm{Lie}\, N_{\widetilde{G}} \subset \mathrm{Lie}\, \widetilde{P},$$

hence by considering the image in the adjoint quotient we get $N_G \subset P$, which is a contradiction by Remark 3.1.4. If $p = 3$ then the above Proposition implies that $\mathrm{Lie}\,\widetilde{P} = \mathrm{Lie}\,\widetilde{G}$, hence the Frobenius kernel satisfies ${}_1\widetilde{G} \subset \widetilde{P}$, and its image in the adjoint quotient is a normal subgroup of G contained in P, which gives again a contradiction. Therefore in both cases P must be a smooth parabolic. □

3.1.3.2. *Reduced parabolic P_m, $m < n$.* Let us consider again a k-vector space V of dimension $2n$ and denote as $\widetilde{G}$ the group $\mathrm{Sp}_{2n} = \mathrm{Sp}(V)$, of type C_n with $n \geq 2$ and k of characteristic $p = 2$ or 3. Its root system has been recalled in (3.1.2). Let us fix an integer $1 \leq m < n$ and consider - keeping the notation recalled at the beginning of this subsection - the maximal reduced parabolic

$$P_m := P^{\alpha_m},$$

associated to the short simple root α_m, which is the subgroup scheme stabilizing an isotropic vector subspace of dimension m: let us denote the latter as W. Then, P_m also stabilizes its orthogonal with respect to the symplectic form on V: denoting as P_m^- the opposite parabolic subgroup and as L_m their common Levi subgroup, one finds

$$\begin{aligned}
P_m &= \mathrm{Stab}(W \subset W^\perp \subset V) = \mathrm{Stab}(W \subset W \oplus U \subset V)\\
P_m^- &= \mathrm{Stab}(W^* \subset (W^*)^\perp \subset V) = \mathrm{Stab}(W \subset W^* \oplus U \subset V)\\
L_m &= P_m \cap P_m^- = \mathrm{GL}(W) \times \mathrm{Sp}(U) \simeq \mathrm{GL}_m \times \mathrm{Sp}_{2n-2m}\,.
\end{aligned}$$

In other words, the choice of such a Levi subgroup corresponds to fixing a vector subspace U satisfying $V = W \oplus U \oplus W^*$. Let us also remark that L has root system Ψ given by

$$\Psi^+ = \{\varepsilon_i - \varepsilon_j,\, i < j \leq m\} \cup \{\varepsilon_i - \varepsilon_j, \varepsilon_i + \varepsilon_j,\, m < i < j\} \cup \{2\varepsilon_j,\, m < j\}.$$

This can be visualized in the following block decomposition :

$$L_m = \left\{ \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & -(A^{-1})^\sharp \end{pmatrix} : A \in \mathrm{GL}(W), B \in \mathrm{Sp}(U) \right\} \subset P_m = \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \right\}$$

Proposition 3.1.10. *Assume given a nonreduced parabolic subgroup P such that $P_{red} = P_m$. Then* $\mathrm{Lie}\,P = \mathrm{Lie}\,\widetilde{G}$ *or* $\mathrm{Lie}\,P = \mathrm{Lie}\,P_m + \mathfrak{g}_<$. *If $p = 3$, then necessarily* $\mathrm{Lie}\,P = \mathrm{Lie}\,\widetilde{G}$.

Proof. The Lie algebra of P_m contains all root subspaces except for those associated to negative roots containing α_m in their support, hence

$$V_m := \mathrm{Lie}\,\widetilde{G}/\,\mathrm{Lie}\,P_m = \left(\bigoplus_{i<j\leq m} \mathfrak{g}_{-\varepsilon_i-\varepsilon_j} \right) \oplus \left(\bigoplus_{j\leq m} \mathfrak{g}_{-2\varepsilon_j} \right) \bigoplus_{i\leq m<j} \left(\mathfrak{g}_{-\varepsilon_i-\varepsilon_j} \oplus \mathfrak{g}_{-\varepsilon_i+\varepsilon_j} \right)$$

More concretely, since $L_m = \mathrm{Stab}(W) \cap \mathrm{Stab}(W^*)$, the Levi subgroup acts on V_m as follows. First, a matrix in

$$\left(\bigoplus_{i<j\leq m} \mathfrak{g}_{-\varepsilon_i-\varepsilon_j}\right) \oplus \left(\bigoplus_{j\leq m} \mathfrak{g}_{-2\varepsilon_j}\right) \tag{3.1.3}$$

is of the form

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \Omega_m X & 0 & 0 \end{pmatrix}$$

with $X \in \mathrm{Sym}^2(W)$, and the L_m-action on it is given by

$$\begin{aligned}(A,B)\cdot X &\simeq \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & -(A^{-1})^\sharp \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ X & 0 & 0 \end{pmatrix}\begin{pmatrix} A^{-1} & 0 & 0 \\ 0 & B^{-1} & 0 \\ 0 & 0 & -A^\sharp \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\Omega_m({}^tA^{-1}XA^{-1}) & 0 & 0 \end{pmatrix} \simeq {}^tA^{-1}XA^{-1},\end{aligned}$$

hence this L_m-module is isomorphic to the dual of the standard representation of GL_m acting on $\mathrm{Sym}^2(k^m)$.

Let us assume that the characteristic of the base field is $p = 2$: then the dual of $\mathrm{Sym}^2(W)$ has an irreducible L_m-quotient given by $\oplus_{j\leq m}\mathfrak{g}_{-2\varepsilon_j}$: this proves that, once such a root subspace is contained in $\mathrm{Lie}\,P$ for some $j \leq m$, then all root subspaces associated to long negative roots are. If $p = 3$, then $\mathrm{Sym}^2(W)^\vee$ is already irreducible itself, hence either all subspaces in (3.1.3) are contained in $\mathrm{Lie}\,P$, or none of them is.
On the other hand, by Remark 3.1.7, an element of the quotient

$$\bigoplus_{i\leq m<j} \left(\mathfrak{g}_{-\varepsilon_i-\varepsilon_j} \oplus \mathfrak{g}_{-\varepsilon_i+\varepsilon_j}\right) =: M$$

is of the form

$$\begin{pmatrix} 0 & 0 & 0 \\ Y & 0 & 0 \\ 0 & Y^\flat & 0 \end{pmatrix}, \quad \text{where } Y^\flat := \Omega_m {}^tY \begin{pmatrix} 0 & \Omega_{n-m} \\ \Omega_{n-m} & 0 \end{pmatrix}$$

with $Y \in \mathrm{Hom}_k(W, U)$. This gives the following L_m-action on M

$$(A,B)\cdot Y \simeq \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & -(A^{-1})^\sharp \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ Y & 0 & 0 \\ 0 & Y^\flat & 0 \end{pmatrix}\begin{pmatrix} A^{-1} & 0 & 0 \\ 0 & B^{-1} & 0 \\ 0 & 0 & -A^\sharp \end{pmatrix} \simeq BYA^{-1},$$

because B being an element of $\mathrm{Sp}(U)$ implies

$$(A^{-1})^\sharp Y^\flat B^{-1} = \Omega_m {}^tA^{-1}\,{}^tY {}^tB \begin{pmatrix} 0 & \Omega_{n-m} \\ -\Omega_{n-m} & 0 \end{pmatrix} = (BYA^{-1})^\flat.$$

Thus, M is isomorphic as an L_m-module to the representation

$$\mathrm{GL}_m \times \mathrm{Sp}_{2n-2m} \curvearrowright \mathrm{Hom}_k(k^m, k^{2n-2m}), \quad (A,B)\cdot Y = BYA^{-1}$$

The latter can be seen (as an L_m-module) as the outer product of the dual of the standard action of GL_m on k^m and of the standard action of Sp_{2n-2m} on k^{2n-2m}. Since both these representations are irreducible, we can conclude that M is an irreducible L_m-module.
Now, let us go back to the parabolic subgroup P: being nonreduced, $\mathrm{Lie}\, P/ \mathrm{Lie}\, P_m$ is a nontrivial L_m-submodule of V_m. We already know that assuming such a quotient to contain some $\mathfrak{g}_{-2\varepsilon_j}$ implies it contains all of them, thus we still need three claims to conclude the proof:

(a) assuming $\mathrm{Lie}\, P/ \mathrm{Lie}\, P_m$ to contain a subspace associated to a long negative root implies it also contains a subspace associated to a short negative root;
(b) assuming it to contain a subspace associated to a short negative root implies it contains all of them;
(c) when $p = 3$, assuming it to contain a subspace associated to a short negative root implies it also contains a subspace associated to a long negative root.

For (a), assume $\mathfrak{g}_{-2\varepsilon_j} \subset \mathrm{Lie}\, P$ for some $j \leq m$, then consider roots

$$\begin{aligned} \gamma &= -2\varepsilon_j, \quad \text{satisfying } X_\gamma \in \mathrm{Lie}\, P \\ \delta &= \varepsilon_j - \varepsilon_n, \quad \text{satisfying } X_\delta \in \mathrm{Lie}\, B \subset \mathrm{Lie}\, P. \end{aligned}$$

Since $\gamma + \delta$ is a root and $\delta - \gamma$ is not, Lemma 3.1.5 yields

$$[X_{-2\varepsilon_j}, X_{\varepsilon_j - \varepsilon_n}] = \pm X_{-\varepsilon_j - \varepsilon_n} \in \mathrm{Lie}\, P.$$

Let us remark that (a) is automatically true when $p = 3$ due to the irreducibility of the L_m-module $\mathrm{Sym}^2(W)$, without needing to consider any structure constant.
For (b), first assume some $\mathfrak{g}_\eta \subset M$ is also contained in $\mathrm{Lie}\, P$. Then $M \subset \mathrm{Lie}\, P$ because of its irreducibility as L_m-quotient of V_m. Moreover, fixing $i < j \leq m$ and applying Lemma 3.1.5 to $\gamma = -\varepsilon_i - \varepsilon_n$ and $\delta = -\varepsilon_j + \varepsilon_n$, satisfying $X_\gamma, X_\delta \in M$, we obtain

$$[X_{-\varepsilon_i - \varepsilon_j}, X_{-\varepsilon_j + \varepsilon_n}] = \pm X_{-\varepsilon_i - \varepsilon_j} \in \mathrm{Lie}\, P.$$

Thus (b) holds in this case. On the other hand, let us start by assuming that $\mathfrak{g}_{-\varepsilon_i - \varepsilon_j} \subset \mathrm{Lie}\, P$ for some $i < j \leq m$. Then, applying Lemma 3.1.5 to $\gamma = -\varepsilon_i - \varepsilon_j$ and $\delta = \varepsilon_j - \varepsilon_n \in \Phi^+$ yields

$$[X_{-\varepsilon_i - \varepsilon_j}, X_{\varepsilon_j - \varepsilon_n}] = \pm X_{-\varepsilon_i - \varepsilon_n} \in \mathrm{Lie}\, P$$

so we conclude that some $\mathfrak{g}_\nu \subset M$ is contained in $\mathrm{Lie}\, P$ and conclude by the beginning of the proof of (b).
For (c) it is enough to use (b) and the irreducibility of $\mathrm{Sym}^2(W)$ when $p = 3$. □

Proof. (**of Theorem 3.1.2 in type C_n when $P_{\mathrm{red}} = P_m$**)
Let G be simple adjoint of type C_n and $X = G/P$ with a faithful G-action such that $P_{\mathrm{red}} = P^{\alpha_m}$ and P is nonreduced. Define $\widetilde{P} \subset \widetilde{G} = \mathrm{Sp}_{2n}$ as being the preimage of P in the simply connected cover: it is a nonreduced parabolic subgroup satisfying $\widetilde{P}_{\mathrm{red}} = P_m$. When $p = 2$, Proposition 3.1.10 implies that

$$\langle\, \mathrm{Lie}(\gamma^\vee(\mathbf{G}_m)) \colon \gamma \in \Phi_< \,\rangle \oplus \mathfrak{g}_< = \mathrm{Lie}\, N_{\widetilde{G}} \subset \mathrm{Lie}\, \widetilde{P},$$

hence by considering the image in the adjoint quotient we get $N_G \subset P$, which is a contradiction by Remark 3.1.4. If $p = 3$ then Proposition 3.1.10 implies that $\mathrm{Lie}\, \widetilde{P} = \mathrm{Lie}\, \widetilde{G}$, hence the Frobenius kernel satisfies ${}_1\widetilde{G} \subset \widetilde{P}$, and its image in the adjoint quotient is a normal subgroup of G contained in P, which gives again a contradiction. Therefore in both cases P must be a smooth parabolic. $\square$

3.1.4. Type B_n. The aim of this subsection is to get the same results for the group of type B_n, with the help of some of the computations involving structure constants, which we have already done in case of type C_n.

3.1.4.1. *Lie algebra of* SO_{2n+1}. Before continuing with our proof, let us compute what $\mathrm{Lie}\, G$ looks like, where $G = \mathrm{SO}_{2n+1} = \mathrm{SO}(k^{2n+1})$ is defined as being relative to the quadratic form

$$Q(x) = x_n^2 + \sum_{i=0}^{n-1} x_i x_{2n-i},$$

in order to determine all its root spaces and be able to make explicit computations with them. To do this, let us consider as maximal torus $T \subset G$ the one given by diagonal matrices of the form

$$t = \mathrm{diag}(t_1, \dots, t_n, 1, t_n^{-1}, \dots, t_1^{-1}) \in \mathrm{GL}_{2n+1}, \tag{3.1.4}$$

while the Borel subgroup is given by upper triangular matrices in G. The Lie algebra is given by all matrices of the form

$$M = \begin{pmatrix} & f_0 & \\ A = (a_{ij})_{i,j=1}^n & \vdots & B = (b_{ij})_{i,j=1}^n \\ & f_{n-1} & \\ g_0 \dots g_{n-1} & h & g_{n+1} \dots g_{2n} \\ & f_{n+1} & \\ C = (c_{ij})_{i,j=1}^n & \vdots & D = (d_{ij})_{i,j=1}^n \\ & f_{2n} & \end{pmatrix} \in \mathfrak{g}_{2n+1}$$

satisfying $Q((1+\epsilon M)x) = Q(x)$ for all $x \in k^{2n+1}$, where $\epsilon^2 = 0$. Let us compute

$$\begin{aligned} Q((1+\epsilon M)x) &= (x_n + \epsilon(g_0 x_0 + \ldots + g_{n-1}x_{n-1} + h x_n + g_{n+1}x_{n+1} + \ldots g_{2n}x_{2n}))^2 \\ &+ \sum_{i=0}^{n-1} \left(x_i + \epsilon \left(\sum_{j=0}^{n-1} a_{ij} x_j + f_i x_n + \sum_{m=0}^{n-1} b_{i,n+1-m} x_{2n-m} \right) \right) \\ &\cdot \left(x_{2n-i} + \epsilon \left(\sum_{r=0}^{n-1} c_{n+1-i,r} x_r + f_{2n-i} x_n + \sum_{l=0}^{n-1} d_{n+1-i,n+1-l} x_{2n-l} \right) \right) \\ &= Q(x) + \epsilon [2h x_n^2 + \sum_{i=0}^{n-1} (f_{2n-i} + 2g_i) x_i x_n + \sum_{i=0}^{n-1} (f_i + 2g_{2n-i}) x_n x_{2n-i} \\ &+ \sum_{i,m=0}^{n-1} b_{i,n+1-m} x_{2n-m} x_{2n-i} + \sum_{i,r=0}^{n-1} c_{n+1-i,r} x_i x_r + \sum_{i,j=0}^{n-1} (a_{ij} + d_{n+1-j,n+1-i}) x_j x_{2n-i}] \end{aligned}$$

Asking the above quantity to be equal to $Q(x)$ gives the following conditions:

$$2h = 0, \quad f_i = -2g_{2n-i}, \quad f_{2n-i} = g_i, \quad D = -A^\sharp, \quad C = -C^\sharp, \quad B = -B^\sharp,$$

where we keep the notation (3.1.1). Moreover, the matrices $\Omega_n B$ and $\Omega_n C$ have zero diagonal. Since the group considered is special orthogonal, the last condition on the determinant implies that the trace of the matrix must be zero hence $h = 0$. The result is thus

$$\operatorname{Lie}\mathrm{SO}_{2n+1} = \left\{ \begin{pmatrix} A & -2\Omega_n w & B \\ {}^t v & 0 & {}^t w \\ C & -2\Omega_n v & -A^\sharp \end{pmatrix} \in \mathfrak{gl}_{2n+1} \colon C = -C^\sharp,\, B = -B^\sharp,\, c_{n+1-i,i} = b_{n+1-i,i} = 0 \right\}$$

Remark 3.1.11. Denoting, analogously to the type C_n, as $\varepsilon_i \in X(T)$ the character $t \mapsto t_i$ for $1 \leq i \leq n$, the root spaces are the following :

$$\mathfrak{g}_{-\varepsilon_i} = k \begin{pmatrix} 0 & 0 & 0 \\ {}^t e_i & 0 & 0 \\ 0 & -2e_{n+1-i} & 0 \end{pmatrix},$$

$$\mathfrak{g}_{\varepsilon_i} = k \begin{pmatrix} 0 & -2e_i & 0 \\ 0 & 0 & {}^t e_{n+1-i} \\ 0 & 0 & 0 \end{pmatrix}, \qquad 1 \leq i \leq n,$$

$$\mathfrak{g}_{\varepsilon_i+\varepsilon_j} = k \begin{pmatrix} 0 & 0 & (E_{ij}+E_{ji})\Omega_n \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\mathfrak{g}_{-\varepsilon_i-\varepsilon_j} = k \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \Omega_n(E_{ij}+E_{ji}) & 0 & 0 \end{pmatrix}, \qquad i < j,$$

$$\mathfrak{g}_{\varepsilon_i-\varepsilon_j} = k \begin{pmatrix} E_{ij} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -E_{ij}^\sharp \end{pmatrix},$$

$$\mathfrak{g}_{-\varepsilon_i+\varepsilon_j} = k \begin{pmatrix} E_{ji} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -E_{ji}^\sharp \end{pmatrix}, \qquad i < j,$$

where e_i denotes the standard basis of k^n and E_{ij} the square matrix of order n with all zero entries except for the (i, j)-th which is equal to one.

We thus verify that the root system $\Phi = \Phi(G, T)$ is given by

$$\Phi^+ = \{\varepsilon_i - \varepsilon_j,\, \varepsilon_i + \varepsilon_j,\, 1 \leq i < j \leq n\} \cup \{\varepsilon_i,\, 1 \leq i \leq n\},$$

with basis Δ consisting of the following roots :

$$\alpha_1 = \varepsilon_1 - \varepsilon_2,\, \dots,\, \alpha_{n-1} = \varepsilon_{n-1} - \varepsilon_n,\, \alpha_n = \varepsilon_n. \tag{3.1.5}$$

3.1.4.2. *Reduced parabolic P_n.* Going back to our setting, let us consider the maximal reduced parabolic subgroup $P_n = P^{\alpha_n}$ associated to the short simple root α_n, i.e. the stabilizer of the isotropic vector subspace $W = ke_0 \oplus \cdots \oplus ke_{n-1} \subset V$ of dimension n, where $G = \mathrm{SO}(V)$ and $(e_i)_{i=0}^{2n}$ denotes the standard basis of k^{2n+1}. Since its Levi subgroup

$L_n = P_n \cap P_n^-$ stabilizes both W and its dual $W^* = ke_{n+1} \oplus \cdots \oplus ke_{2n}$, we conclude that it is of the form

$$L_n = \left\{ \begin{pmatrix} A & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & (A^{-1})^\sharp \end{pmatrix} : A \in \mathrm{GL}(W) \simeq \mathrm{GL}_n \right\} \subset P_n = \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \right\} \subset G,$$

where $V = W \oplus ke_n \oplus W^*$. In particular, L_n is isomorphic to GL_n, with root system Ψ given by

$$\Psi^+ = \{\varepsilon_i - \varepsilon_j,\, 1 \leq i < j \leq n\}.$$

Proposition 3.1.12. *Assume given a nonreduced parabolic subgroup P such that $P_{red} = P_n$. Then* $\operatorname{Lie} P = \operatorname{Lie} G$ *or* $\operatorname{Lie} P = \operatorname{Lie} P_n + \mathfrak{g}_<$. *If $p = 3$, then necessarily* $\operatorname{Lie} P = \operatorname{Lie} G$.

Proof. First, by definition of P_n its Lie algebra is given by

$$\operatorname{Lie} P_n = \operatorname{Lie} L_n \bigoplus_{i<j} \mathfrak{g}_{\varepsilon_i+\varepsilon_j} \bigoplus_i \mathfrak{g}_{\varepsilon_i},$$

Since P is assumed to be nonreduced, $\operatorname{Lie} P_n \subsetneq \operatorname{Lie} P$ hence :

(1) either there is some i such that $\mathfrak{g}_{-\varepsilon_i} \subset \operatorname{Lie} P$,
(2) or there is some $i < j$ such that $\mathfrak{g}_{-\varepsilon_i-\varepsilon_j} \subset \operatorname{Lie} P$.

Let us start by assuming (1) and fix such an index i. To show that all other $\mathfrak{g}_{-\varepsilon_j}$ are then contained in $\operatorname{Lie} P$, let us consider the L_n-module

$$V_n := \operatorname{Lie} G / \operatorname{Lie} P_n = \left(\bigoplus_{i<j} \mathfrak{g}_{-\varepsilon_i-\varepsilon_j} \right) \oplus \left(\bigoplus_i \mathfrak{g}_{-\varepsilon_i} \right).$$

By Remark 3.1.11, a matrix in $\bigoplus_{i=1}^n \mathfrak{g}_{-\varepsilon_i}$ is of the form

$$\begin{pmatrix} 0 & 0 & 0 \\ {}^t v & 0 & 0 \\ 0 & -2\Omega_n v & 0 \end{pmatrix}$$

for $v \in k^n$, and the dual L_n-action on it is given by

$$({3.1.6}) \qquad {}^tA^{-1} \cdot v = \begin{pmatrix} {}^tA^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & {}^tA^\sharp \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ {}^t v & 0 & 0 \\ 0 & -2\Omega_n v & 0 \end{pmatrix} \begin{pmatrix} {}^tA & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & ({}^tA^{-1})^\sharp \end{pmatrix}$$

$$({3.1.7}) \qquad = \begin{pmatrix} 0 & 0 & 0 \\ {}^t(Av) & 0 & 0 \\ 0 & -2\Omega_n Av & 0 \end{pmatrix} \simeq Av$$

In particular, $\bigoplus_{i=1}^n \mathfrak{g}_{-\varepsilon_i}$ is a simple L_n-submodule of V_n, isomorphic to the dual of the standard representation of GL_n on k^n. Thus, if a root subspace associated to some $-\varepsilon_i$ is contained in $\operatorname{Lie} P$, all of the $\mathfrak{g}_{-\varepsilon_j}$ are too.
Let us assume instead that (2) holds: then, by repeating the same exact reasoning done in case (b) of the preceding subsection, we show that $\operatorname{Lie} P$ contains all weight spaces

associated to long roots. This is due to the fact that the argument above only involves roots of the form $\pm(\varepsilon_l \pm \varepsilon_m)$. Moreover, assume $i \neq n$ and consider roots

$$\gamma = \varepsilon_n, \quad \text{satisfying } X_\gamma \in \operatorname{Lie} L_n \subset \operatorname{Lie} P$$
$$\delta = -\varepsilon_i - \varepsilon_n, \quad \text{satisfying } X_\delta \in \operatorname{Lie} P \text{ by our last assumption.}$$

Thus, $\gamma + \delta = -\varepsilon_i$ is still a root while $\delta - \gamma = -\varepsilon_i - 2\varepsilon_n$ is not: applying Lemma 3.1.5 gives

$$[X_{\varepsilon_n}, X_{-\varepsilon_i-\varepsilon_n}] = \pm X_{-\varepsilon_i} \in \operatorname{Lie} P.$$

In conclusion, when $p = 2$ we have shown that condition (2) implies $\operatorname{Lie} P = \operatorname{Lie} G$, while assuming condition (1) to be true and (2) to be false implies $\operatorname{Lie} P = \operatorname{Lie} P_n + \mathfrak{g}_<$.
If $p = 3$ then the above reasoning still holds; the only remark that we need to add is that $\mathfrak{g}_< \subset \operatorname{Lie} P$ implies that there is a long negative root ν satisfying $\mathfrak{g}_\nu \subset \operatorname{Lie} P / \operatorname{Lie} P_n$. For this, let us consider roots

$$\gamma = -\varepsilon_1 \text{ and } \delta = -\varepsilon_n, \text{ satisfying } X_\gamma, X_\delta \in \operatorname{Lie} P \text{ by our last assumption.}$$

Thus, $\gamma + \delta = -\varepsilon_1 - \varepsilon_n$ is still a root, $\gamma - \delta = -\varepsilon_1 + \varepsilon_n$ is too, while $\gamma - 2\delta = -\varepsilon_1 + 2\varepsilon_n$ is not: applying Lemma 3.1.5 gives

$$[X_{-\varepsilon_1}, X_{-\varepsilon_n}] = \pm 2X_{-\varepsilon_1-\varepsilon_n}, \quad \text{hence } X_{-\varepsilon_1-\varepsilon_n} \in \operatorname{Lie} P.$$

Clearly, this last step of the proof would not work under the hypothesis $p = 2$. □

PROOF. (**of Theorem 3.1.2 in type B_n when $P_{\text{red}} = P_n$**)
Let G be simple adjoint of type B_n and $X = G/P$ with a faithful G-action such that $P_{\text{red}} = P_n = P^{\alpha_n}$ and P is nonreduced. When $p = 2$, the above Proposition, together with the computation of Example 2.5.16, imply that

$$\mathfrak{g}_< = \operatorname{Lie} N_G \subset \operatorname{Lie} P,$$

hence we get $N_G \subset P$, which is a contradiction by Remark 3.1.4. When $p = 3$, the above Proposition implies that $\operatorname{Lie} P = \operatorname{Lie} G$, hence the Frobenius kernel satisfies ${}_1G \subset P$, which gives again a contradiction. Therefore in both cases P must be a smooth parabolic. □

Remark 3.1.13. A small additional remark is needed in order to have a uniform statement later on, since this is the only case where the group G is not simply connected: let $\psi\colon \widetilde{G} = \operatorname{Spin}_{2n+1} \longrightarrow G = \operatorname{SO}_{2n+1}$ be the quotient morphism and consider a nonreduced parabolic subgroup $P \subset \widetilde{G}$ such that $P_{\text{red}} = P^{\alpha_n}$. The above reasoning implies that $\psi(P)$ either contains N_G - when such a subgroup is defined - or it contains the Frobenius kernel ${}_1G$. In particular, P contains a normal noncentral subgroup of height one, namely $P \cap \psi^{-1}(N_G)$ or $P \cap \psi^{-1}({}_1G)$.

3.1.4.3. *Reduced parabolic P_m, $m < n$.* Let us consider again a k-vector space V of dimension $2n+1$ and denote as G the group $\operatorname{SO}_{2n+1} = \operatorname{SO}(V)$, of type B_n with $n \geq 2$ and k of characteristic $p = 2$ or 3. Moreover, let us consider the maximal reduced parabolic subgroup

$$P_m := P^{\alpha_m}$$

associated to a long simple root α_m for some $m < n$, keeping notations from (3.1.5). This subgroup is the stabilizer of an isotropic vector subspace $W = ke_0 \oplus \cdots \oplus ke_{m-1} \subset V$ of dimension m, where $(e_i)_{i=0}^{2n}$ denotes the standard basis of k^{2n+1}. Since its Levi subgroup $L_m = P_m \cap P_m^-$ stabilizes both W and its dual $W^* = ke_{2n-m+1} \oplus \cdots \oplus ke_{2n}$, we conclude that it is of the form

$$L_m = \left\{ \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & (A^{-1})^\sharp \end{pmatrix} : A \in \mathrm{GL}(W), B \in \mathrm{SO}(U) \right\} \subset P_m = \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \right\}$$

where $V = W \oplus U \oplus W^*$. In particular, $L_m \simeq \mathrm{GL}_m \times \mathrm{SO}_{2n-2m+1}$ with root system Ψ given by

$$\Psi^+ = \{\varepsilon_i - \varepsilon_j,\, i < j \leq m\} \cup \{\varepsilon_i - \varepsilon_j, \varepsilon_i + \varepsilon_j,\, m < i < j\} \cup \{\varepsilon_j,\, m < j\}.$$

Proposition 3.1.14. *Assume given a nonreduced parabolic subgroup P such that $P_{red} = P_m$. Then* $\operatorname{Lie} P = \operatorname{Lie} G$ *or* $\operatorname{Lie} P = \operatorname{Lie} P_m + \mathfrak{g}_<$. *If $p = 3$, then necessarily* $\operatorname{Lie} P = \operatorname{Lie} G$.

Proof. The Lie algebra of P_m contains all root subspaces except for those associated to negative roots containing α_m in their support, hence

$$V_m := \operatorname{Lie} G / \operatorname{Lie} P_m = \left(\bigoplus_{i<j\leq m} \mathfrak{g}_{-\varepsilon_i - \varepsilon_j} \right) \oplus \left(\bigoplus_{j \leq m} \mathfrak{g}_{-\varepsilon_j} \right) \bigoplus_{i \leq m < j} \left(\mathfrak{g}_{-\varepsilon_i - \varepsilon_j} \oplus \mathfrak{g}_{-\varepsilon_i + \varepsilon_j} \right)$$

The analogous computations as those in the proofs of Proposition 3.1.10 and (3.1.6) imply that, as L_m-modules,

(1) $\bigoplus_{j\leq m} \mathfrak{g}_{-\varepsilon_j}$ is isomorphic to the dual of the standard representation of GL_n on k^m, hence it is in particular a simple L_m-quotient of V_m ;
(2) $\bigoplus_{i\leq m<j} \left(\mathfrak{g}_{-\varepsilon_i - \varepsilon_j} \oplus \mathfrak{g}_{-\varepsilon_i + \varepsilon_j} \right)$ is isomorphic to the following representation, which gives a second irreducible L_m-quotient of V_m :

$$\mathrm{GL}_m \times \mathrm{SO}_{2n-2m+1} \curvearrowright \mathrm{Hom}_k(k^m, k^{2n-2m+1}), \quad (A, B) \cdot Y = BYA^{-1}.$$

Now, first assume $\mathfrak{g}_{-\varepsilon_l} \subset \operatorname{Lie} P$ for some $l \leq m$. Then $\bigoplus_{j\leq m} \mathfrak{g}_{-\varepsilon_j}$ is contained in $\operatorname{Lie} P$, since $\operatorname{Lie} P / \operatorname{Lie} P_m$ is a nontrivial L_m-submodule of V_m. Hence in this case $\mathfrak{g}_< \subset \operatorname{Lie} P$.
The only other possibility is to start by assuming that $\mathfrak{g}_\gamma \subset \operatorname{Lie} P$ for some long negative root γ containing α_m in its support. Then one can repeat the same exact reasoning of point (b) in the proof of Proposition 3.1.10, since it involves only roots of the form $\pm(\varepsilon_l \pm \varepsilon_m)$ with $l < m$, to conclude that all root subspaces associated to long negative roots are also contained in $\operatorname{Lie} P$. To conclude that, in this case, $\operatorname{Lie} P = \operatorname{Lie} G$, it suffices to apply Lemma 3.1.5 to $\gamma = -\varepsilon_1 - \varepsilon_m$ and $\delta = \varepsilon_m$, which gives

$$[X_{-\varepsilon_1 - \varepsilon_m}, X_{\varepsilon_m}] = \pm X_{-\varepsilon_1} \in \operatorname{Lie} P$$

as wanted.
Up to this point everything holds in both characteristic $p = 2$ and 3. To conclude it is enough to show that, when $p = 3$, if $\mathfrak{g}_< \subset \operatorname{Lie} P$ then there is a long negative root ν satisfying $\mathfrak{g}_\nu \subset \operatorname{Lie} P / \operatorname{Lie} P_m$. For this, let us consider roots

$$\gamma = -\varepsilon_1 \text{ and } \delta = -\varepsilon_n, \text{ satisfying } X_\gamma, X_\delta \in \operatorname{Lie} P \text{ by our last assumption.}$$

Thus, $\gamma+\delta=-\varepsilon_1-\varepsilon_n$ is still a root, $\gamma-\delta=-\varepsilon_1+\varepsilon_n$ is too, while $\gamma-2\delta=-\varepsilon_1+2\varepsilon_n$ is not: applying Lemma 3.1.5 gives

$$[X_{-\varepsilon_1}, X_{-\varepsilon_n}] = \pm 2X_{-\varepsilon_1-\varepsilon_n}, \quad \text{hence } X_{-\varepsilon_1-\varepsilon_n} \in \operatorname{Lie} P$$

as wanted. □

PROOF. (**of Theorem 3.1.2 in type B_n when $P_{\mathrm{red}} = P_m$**)
Let G be simple adjoint of type B_n and $X = G/P$ with a faithful G-action such that $P_{\mathrm{red}} = P^{\alpha_m}$ and P is nonreduced. When $p = 2$ the above Proposition, together with Example 2.5.16, imply that

$$\mathfrak{g}_< = \operatorname{Lie} N_G \subset \operatorname{Lie} P,$$

hence we get $N_G \subset P$, which is a contradiction by Remark 3.1.4. When $p = 3$, the above Proposition implies that $\operatorname{Lie} P = \operatorname{Lie} G$, hence the Frobenius kernel satisfies $G_1 \subset P$, which gives again a contradiction. Therefore in both cases P must be a smooth parabolic. □

Remark 3.1.15. As in Remark 3.1.13 above, we can conclude that if $P \subset \operatorname{Spin}_{2n+1}$ is a nonreduced parabolic subgroup satisfying $P_{\mathrm{red}} = P^{\alpha_m}$, then it contains a normal noncentral subgroup of height one.

3.1.5. Type F_4. Let us consider a simple group G with root system F_4 over an algebraically closed field k of characteristic $p = 2$ or 3. Following notations from [**Bou**], a basis Δ of its root system Φ is given by

$$\alpha_1 = \varepsilon_2 - \varepsilon_3, \quad \alpha_2 = \varepsilon_3 - \varepsilon_4, \quad \alpha_3 = \varepsilon_4, \quad \alpha_4 = \frac{1}{2}(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4),$$

satisfying the relations

$$||\alpha_1||^2 = ||\alpha_2||^2 = 2, \quad ||\alpha_3||^2 = ||\alpha_4||^2 = 1$$

and

$$(\alpha_1,\alpha_2) = (\alpha_2,\alpha_3) = -1, \quad (\alpha_1,\alpha_3) = (\alpha_1,\alpha_4) = (\alpha_2,\alpha_4) = 0, \quad (\alpha_3,\alpha_4) = -\frac{1}{2}. \tag{3.1.8}$$

Let us denote the associated maximal reduced parabolic subgroups as $P_i := P^{\alpha_i}$, for $i \in \{1,2,3,4\}$. Let us also recall that, when $p = 2$, $N_G \subset G$ is the unique subgroup of height one such that

$$\operatorname{Lie} N_G = \operatorname{Lie} \alpha_3^\vee(\mathbf{G}_m) \oplus \operatorname{Lie} \alpha_4^\vee(\mathbf{G}_m) \oplus \mathfrak{g}_<,$$

where the short positive roots are

$$\begin{gathered} \alpha_3,\ \alpha_4,\ \alpha_2+\alpha_3,\ \alpha_3+\alpha_4,\ \alpha_1+\alpha_2+\alpha_3,\ \alpha_2+2\alpha_3+\alpha_4, \\ \alpha_2+\alpha_3+\alpha_4,\ \alpha_1+2\alpha_2+3\alpha_3+2\alpha_4,\ \alpha_1+\alpha_2+\alpha_3+\alpha_4, \\ \alpha_1+\alpha_2+2\alpha_3+\alpha_4,\ \alpha_1+2\alpha_2+2\alpha_3+\alpha_4,\ \alpha_1+2\alpha_2+3\alpha_3+\alpha_4. \end{gathered}$$

Proposition 3.1.16. *Assume given a nonreduced parabolic subgroup P such that $P_{red} = P_i$ for some i. Then $\operatorname{Lie} P = \operatorname{Lie} G$ or $\operatorname{Lie} P = \operatorname{Lie} P_i + \mathfrak{g}_<$. If $p = 3$, then necessarily $\operatorname{Lie} P = \operatorname{Lie} G$.*

PROOF. Before starting a case-by-case analysis, let us denote as s_i, for $i = 1,2,3,4$, the reflection associated to the simple root α_i, i.e.

$$s_i(\gamma) = \gamma - 2\frac{(\alpha_i,\gamma)}{(\alpha_i,\alpha_i)}\alpha_i, \qquad \text{for all } \gamma \in \Phi. \tag{3.1.9}$$

Case $P_{\mathbf{red}} = P_1$.

Let us assume that $P_{\text{red}} = P_1$ and denote as $L_1 := P_1 \cap P_1^-$ the Levi subgroup: its root system is of type C_3 with basis consisting of short roots α_4, α_3 and the long root α_2. Moreover, L_1 acts on the vector space

$$V_1 := \operatorname{Lie} G/\operatorname{Lie} P_1 = \bigoplus_{\gamma\in\Gamma_1} \mathfrak{g}_{-\gamma},$$

where Γ_1 is the subset of all positive roots satisfying $\alpha_1 \in \operatorname{Supp}(\gamma)$. As usual, let us consider the nonzero vector subspace $W_1 := \operatorname{Lie} P/\operatorname{Lie} P_1$, which is a L_1-submodule of V_1: the set of its weights, which we denote Ω_1, must be stable under the reflections s_2, s_3 and s_4. Our aim is to show that

$$\text{either} \quad \Omega_1 = \Gamma_1 \cap \Phi_< \quad \text{or} \quad \Omega_1 = \Gamma_1 : \tag{3.1.10}$$

in other words, either $W_1 = \oplus_{\gamma\in\Gamma_1\cap\Phi_<}\mathfrak{g}_{-\gamma}$ or $W_1 = V_1$.
First, let us show that the Weyl group $W(L_1,T) = \langle s_2, s_3, s_4\rangle$ acts transitively on

$$\begin{aligned}\Gamma_1 \cap \Phi_< = \{&\alpha_1+\alpha_2+\alpha_3,\ \alpha_1+\alpha_2+\alpha_3+\alpha_4,\ \alpha_1+\alpha_2+2\alpha_3+\alpha_4,\ \alpha_1+2\alpha_2+2\alpha_3+\alpha_4,\\ &\alpha_1+2\alpha_2+3\alpha_3+\alpha_4,\ \alpha_1+2\alpha_2+3\alpha_3+2\alpha_4\} :\end{aligned}$$

this implies that either $\Gamma_1 \cap \Phi_< \subset \Omega_1$ or $(\Gamma_1 \cap \Phi_<) \cap \Omega_1 = \emptyset$. The following computations follow directly from (3.1.8) and (3.1.9) :

$$\begin{aligned}
s_4(\alpha_1+\alpha_2+\alpha_3) &= \alpha_1+\alpha_2+\alpha_3+\alpha_4,\\
s_3(\alpha_1+\alpha_2+\alpha_3+\alpha_4) &= \alpha_1+\alpha_2+2\alpha_3+\alpha_4,\\
s_2(\alpha_1+\alpha_2+2\alpha_3+\alpha_4) &= \alpha_1+2\alpha_2+2\alpha_3+\alpha_4,\\
s_3(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4) &= \alpha_1+2\alpha_2+3\alpha_3+\alpha_4,\\
s_4(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4) &= \alpha_1+2\alpha_2+3\alpha_3+2\alpha_4.
\end{aligned}$$

Next, let us show that $W(L_1,T)$ acts transitively on

$$\begin{aligned}(\Gamma_1\cap\Phi_>)\backslash\{\widetilde{\alpha}\} = \{&\alpha_1,\ \alpha_1+\alpha_2,\ \alpha_1+\alpha_2+2\alpha_3,\ \alpha_1+2\alpha_2+2\alpha_3,\ \alpha_1+\alpha_2+2\alpha_3+2\alpha_4,\\ &\alpha_1+2\alpha_2+2\alpha_3+2\alpha_4,\ \alpha_1+2\alpha_2+4\alpha_3+2\alpha_4,\ \alpha_1+3\alpha_2+4\alpha_3+2\alpha_4\},\end{aligned}$$

where $\widetilde{\alpha} := 2\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4$ is the highest root. Let us remark that $\widetilde{\alpha}$ is indeed fixed by the Weyl group of L_1: this is due to the fact that it is the only root whose coefficient of

α_1 is 2 instead of 1. Again, the transitivity of the action is proved by direct computation:

$$\begin{aligned}
&s_2(\alpha_1) = \alpha_1 + \alpha_2,\\
&s_3(\alpha_1 + \alpha_2) = \alpha_1 + \alpha_2 + 2\alpha_3,\\
&s_2(\alpha_1 + \alpha_2 + 2\alpha_3) = \alpha_1 + 2\alpha_2 + 2\alpha_3,\\
&s_4(\alpha_1 + 2\alpha_2 + 2\alpha_3) = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4,\\
&s_3(\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4) = \alpha_1 + 2\alpha_2 + 4\alpha_3 + 2\alpha_4,\\
&s_1(\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4) = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4,\\
&s_2(\alpha_1 + 2\alpha_2 + 4\alpha_3 + 2\alpha_4) = \alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4.
\end{aligned}$$

Thus, either $(\Gamma_1 \cap \Phi_>)\backslash\{\widetilde{\alpha}\} \subset \Omega_1$ or $((\Gamma_1 \cap \Phi_>)\backslash\{\widetilde{\alpha}\}) \cap \Omega_1 = \emptyset$. Next, we show that $\widetilde{\alpha} \in \Omega_1$ if and only if $(\Gamma_1 \cap \Phi_>)\backslash\{\widetilde{\alpha}\} \subset \Omega_1$.

- Assume that $\mathfrak{g}_{-\widetilde{\alpha}} \subset W_1$. Then applying Lemma 3.1.5 to $\gamma = -\widetilde{\alpha}$ and $\delta = \alpha_1 + 2\alpha_2 + 2\alpha_3$ gives
$$[X_{-\widetilde{\alpha}}, X_{\alpha_1+2\alpha_2+2\alpha_3}] = \pm X_{-\alpha_1-\alpha_2-2\alpha_3-2\alpha_4} \in \operatorname{Lie} P,$$
since $\gamma + \delta$ is a root while $\gamma - \delta = -3\alpha_1 - 5\alpha_2 - 6\alpha_3 - 2\alpha_4$ is not. This implies that the long root $\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4$ belongs to Ω_1.
- Assume that $(\Gamma_1 \cap \Phi_>)\backslash\{\widetilde{\alpha}\} \subset \Omega_1$. In particular,
$$\mathfrak{g}_{-\alpha_1-2\alpha_2-2\alpha_3} \oplus \mathfrak{g}_{-\alpha_1-\alpha_2-2\alpha_3-2\alpha_4} \subset \operatorname{Lie} P.$$
Thus, we can apply Lemma 3.1.5 to $\gamma = -\alpha_1 - 2\alpha_2 - 2\alpha_3$ and $\delta = -\alpha_1 - \alpha_2 - 2\alpha_3 - 2\alpha_4$ to get
$$[X_{-\alpha_1-2\alpha_2-2\alpha_3}, X_{-\alpha_1-\alpha_2-2\alpha_3-2\alpha_4}] = \pm X_{-\widetilde{\alpha}} \in \operatorname{Lie} P,$$
since $\gamma + \delta$ is a root while $\gamma - \delta = -\alpha_2 + 2\alpha_4$ is not.

The last step in order to prove (3.1.10) consists in showing that $(\Gamma_1 \cap \Phi_>) \subset \Omega_1$ implies $(\Gamma_1 \cap \Phi_<) \cap \Omega_1 \neq \emptyset$ which, by the above reasoning, means $\Gamma_1 = \Omega_1$. By our assumption, the long root $\gamma = -\alpha_1 - 2\alpha_2 - 2\alpha_3 - 2\alpha_4$ satisfies $\mathfrak{g}_\gamma \subset \operatorname{Lie} P$. Setting $\delta = -\alpha_3$ and applying Lemma 3.1.5 gives

$$[X_{-\alpha_1-2\alpha_2-2\alpha_3-2\alpha_4}, X_{-\alpha_3}] = \pm X_{-\alpha_1-2\alpha_2-3\alpha_3-2\alpha_4} \in \operatorname{Lie} P,$$

since $\gamma + \delta$ is a root while $\gamma - \delta = -\alpha_1 - 2\alpha_2 - \alpha_3 - 2\alpha_4$ is not. This concludes the first case.

Case $P_{\mathrm{red}} = P_2$.

Let us assume that $P_{\mathrm{red}} = P_2$ and fix the analogous notation as above: $L_2 := P_2 \cap P_2^-$ acts on

$$W_2 := \operatorname{Lie} P / \operatorname{Lie} P_2 = \bigoplus_{\gamma \in \Omega_2} \mathfrak{g}_{-\gamma} \subset V_2 := \operatorname{Lie} G / \operatorname{Lie} P_2 = \bigoplus_{\gamma \in \Gamma_2} \mathfrak{g}_{-\gamma}$$

and its set of weights Ω_2 must be stable under the action of the Weyl group $W(L_2, T) = \langle s_1, s_3, s_4 \rangle$. Our aim is to show that

$$\text{either} \quad \Omega_2 = \Gamma_2 \cap \Phi_< \quad \text{or} \quad \Omega_2 = \Gamma_2. \tag{3.1.11}$$

First, let us consider the partition of Γ_2 as disjoint union of the following subsets :

$$\Sigma_1 := \{\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4,\ \widetilde{\alpha}\},$$
$$\Sigma_2 := \{\alpha_1 + 2\alpha_2 + 2\alpha_3,\ \alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4,\ \alpha_1 + 2\alpha_2 + 4\alpha_3 + 2\alpha_4\},$$
$$\Sigma_3 := \{\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4,\ \alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4,\ \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4\},$$
$$\Sigma_4 := \{\alpha_2 + \alpha_3,\ \alpha_1 + \alpha_2 + \alpha_3,\ \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4,\ \alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4,$$
$$\alpha_2 + 2\alpha_3 + \alpha_4,\ \alpha_2 + \alpha_3 + \alpha_4\},$$
$$\Sigma_5 := \{\alpha_2 + 2\alpha_3 + \alpha_4,\ \alpha_2 + 2\alpha_3,\ \alpha_1 + \alpha_2,\ \alpha_2,\ \alpha_1 + \alpha_2 + 2\alpha_3,\ \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4\}.$$

Notice that $\Sigma_1 \cup \Sigma_2 \cup \Sigma_5 = \Gamma_2 \cap \Phi_>$ and $\Sigma_3 \cup \Sigma_4 = \Gamma_2 \cap \Phi_<$, so the root lengths once again come into play. Moreover, Σ_1, $\Sigma_2 \cup \Sigma_3$ and $\Sigma_4 \cup \Sigma_5$ are indeed stable under the action of $W(L_2, T)$, since their elements have coefficient 3, 2 and 1 respectively with respect to the simple root α_2. Now, the following computations prove that :

- Σ_1 is stable by $W(L_2, T)$:
$$s_1(\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4) = \widetilde{\alpha};$$
- Σ_2 is stable by $W(L_2, T)$:
$$s_4(\alpha_1 + 2\alpha_2 + 2\alpha_3) = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4,$$
$$s_3(\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4) = \alpha_1 + 2\alpha_2 + 4\alpha_3 + 2\alpha_4;$$
- Σ_3 is stable by $W(L_2, T)$:
$$s_3(\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4) = \alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4,$$
$$s_4(\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4) = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4;$$
- Σ_4 is stable by $W(L_2, T)$:
$$s_1(\alpha_2 + \alpha_3) = \alpha_1 + \alpha_2 + \alpha_3,$$
$$s_4(\alpha_1 + \alpha_2 + \alpha_3) = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4,$$
$$s_1(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) = \alpha_2 + \alpha_3 + \alpha_4,$$
$$s_3(\alpha_2 + \alpha_3 + \alpha_4) = \alpha_2 + 2\alpha_3 + \alpha_4,$$
$$s_1(\alpha_2 + 2\alpha_3 + \alpha_4) = \alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4;$$
- Σ_5 is stable by $W(L_2, T)$:
$$s_4(\alpha_2 + 2\alpha_3 + 2\alpha_4) = \alpha_2 + 2\alpha_3,$$
$$s_3(\alpha_2 + 2\alpha_3) = \alpha_1 + \alpha_2,$$
$$s_1(\alpha_1 + \alpha_2) = \alpha_2 \quad \text{and} \quad s_3(\alpha_1 + \alpha_2) = \alpha_1 + \alpha_2 + 2\alpha_3,$$
$$s_4(\alpha_1 + \alpha_2 + 2\alpha_3) = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4.$$

Thus, for $j = 1, \ldots, 5$, we have shown that $\Sigma_j \cap \Omega_2 \neq \emptyset$ implies that $\Sigma_j \subset \Omega_2$. Next, we prove the following claims by using Lemma 3.1.5 on structure constants :

(a) $\Sigma_1 \subset \Omega_2$ implies that $\Sigma_2 \subset \Omega_2$,
(b) $\Sigma_2 \subset \Omega_2$ implies that $\Sigma_5 \subset \Omega_2$,
(c) $\Sigma_5 \subset \Omega_2$ implies that $\Sigma_2 \subset \Omega_2$,
(d) $\Sigma_2 \cup \Sigma_5 \subset \Omega_2$ implies that $\Sigma_1 \subset \Omega_2$,

(e) $\Sigma_3 \subset \Omega_2$ implies that $\Sigma_4 \subset \Omega_2$,
(f) $\Sigma_4 \subset \Omega_2$ implies that $\Sigma_3 \subset \Omega_2$,
(g) $\Sigma_2 \subset \Omega_2$ implies that $\Sigma_3 \subset \Omega_2$.

The parabolic subgroup P being non-reduced by assumption, the set Ω_2 is nonempty hence, once these implications are proved, it must be either all of Γ_2 or $\Sigma_3 \cup \Sigma_4 = \Gamma_2 \cap \Phi_<$, which proves (3.1.11).

(a): By assumption $\mathfrak{g}_{-\alpha_1-3\alpha_2-4\alpha_3-2\alpha_4} \subset \operatorname{Lie} P$. Set $\gamma = -\alpha_1 - 3\alpha_2 - 4\alpha_3 - 2\alpha_4$ and $\delta = \alpha_2$, then $\gamma - \delta = -\alpha_1 - 4\alpha_2 - 4\alpha_3 - 2\alpha_4$ is not a root hence

$$[X_\gamma, X_\delta] = \pm X_{-\alpha_1-2\alpha_2-4\alpha_3-2\alpha_4} \in \operatorname{Lie} P$$

so $\alpha_1 + 2\alpha_2 + 4\alpha_3 + 2\alpha_4 \in \Sigma_2 \cap \Omega_2$.

(b): By assumption $\mathfrak{g}_{-\alpha_1-2\alpha_2-4\alpha_3-2\alpha_4} \subset \operatorname{Lie} P$. Set $\gamma = -\alpha_1 - 2\alpha_2 - 4\alpha_3 - 2\alpha_4$ and $\delta = \alpha_2 + 2\alpha_3$, then $\gamma - \delta = -\alpha_1 - 3\alpha_2 - 6\alpha_3 - 2\alpha_4$ is not a root hence

$$[X_\gamma, X_\delta] = \pm X_{-\alpha_1-\alpha_2-2\alpha_3-2\alpha_4} \in \operatorname{Lie} P$$

so $\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 \in \Sigma_5 \cap \Omega_2$.

(c): By assumption $\mathfrak{g}_{-\alpha_1-\alpha_2} \oplus \mathfrak{g}_{-\alpha_2-2\alpha_3} \subset \operatorname{Lie} P$. Set $\gamma = -\alpha_1 - \alpha_2$ and $\delta = -\alpha_2 - 2\alpha_3$, then $\gamma - \delta = -\alpha_1 - 2\alpha_3$ is not a root hence

$$[X_\gamma, X_\delta] = \pm X_{-\alpha_1-2\alpha_2-2\alpha_3} \in \operatorname{Lie} P$$

so $\alpha_1 + 2\alpha_2 + 2\alpha_3 \in \Sigma_2 \cap \Omega_2$.

(d): By assumption $\mathfrak{g}_{-\alpha_1-\alpha_2-2\alpha_3-2\alpha_4} \oplus \mathfrak{g}_{-\alpha_1-2\alpha_2-2\alpha_3} \subset \operatorname{Lie} P$. Set $\gamma = -\alpha_1 - \alpha_2 - 2\alpha_3 - 2\alpha_4$ and $\delta = -\alpha_1 - 2\alpha_2 - 2\alpha_3$, then $\gamma - \delta = \alpha_2 - 2\alpha_4$ is not a root hence

$$[X_\gamma, X_\delta] = \pm X_{-\widetilde{\alpha}} \in \operatorname{Lie} P$$

so $\widetilde{\alpha} \in \Sigma_1 \cap \Omega_2$.

(e): By assumption $\mathfrak{g}_{-\alpha_1-2\alpha_2-2\alpha_3-\alpha_4} \subset \operatorname{Lie} P$. Set $\gamma = -\alpha_1 - 2\alpha_2 - 2\alpha_3 - \alpha_4$ and $\delta = \alpha_2$, then $\gamma - \delta = -\alpha_1 - 3\alpha_2 - 2\alpha_3 - \alpha_4$ is not a root hence

$$[X_\gamma, X_\delta] = \pm X_{-\alpha_1-\alpha_2-2\alpha_3-\alpha_4} \in \operatorname{Lie} P$$

so $\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4 \in \Sigma_4 \cap \Omega_2$.

(f): By assumption $\mathfrak{g}_{-\alpha_1-\alpha_2-\alpha_3-\alpha_4} \oplus \mathfrak{g}_{-\alpha_2-2\alpha_3-\alpha_4} \subset \operatorname{Lie} P$. Set $\gamma = -\alpha_1 - \alpha_2 - \alpha_3 - \alpha_4$ and $\delta = -\alpha_2 - 2\alpha_3 - \alpha_4$, then $\gamma - \delta = -\alpha_1 + \alpha_3$ is not a root hence

$$[X_\gamma, X_\delta] = \pm X_{-\alpha_1-2\alpha_2-3\alpha_3-2\alpha_4} \in \operatorname{Lie} P$$

so $\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 \in \Sigma_3 \cap \Omega_2$.

(g): By assumption $\mathfrak{g}_{-\alpha_1-2\alpha_2-2\alpha_3-2\alpha_4} \subset \operatorname{Lie} P$. Set $\gamma = -\alpha_1 - 2\alpha_2 - 2\alpha_3 - 2\alpha_4$ and $\delta = -\alpha_3$, then $\gamma - \delta = -\alpha_1 - 2\alpha_2 - \alpha_3 - 2\alpha_4$ is not a root hence

$$[X_\gamma, X_\delta] = \pm X_{-\alpha_1-2\alpha_2-3\alpha_3-2\alpha_4} \in \operatorname{Lie} P$$

so $\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 \in \Sigma_3 \cap \Omega_2$.

Case $P_{\mathbf{red}} = P_3$.

Let us assume that $P_{\text{red}} = P_3$ and fix the analogous notation as above: $L_3 := P_3 \cap P_3^-$

acts on

$$W_3 := \operatorname{Lie} P/\operatorname{Lie} P_3 = \bigoplus_{\gamma\in\Omega_3} \mathfrak{g}_{-\gamma} \subset V_3 := \operatorname{Lie} G/\operatorname{Lie} P_3 = \bigoplus_{\gamma\in\Gamma_3} \mathfrak{g}_{-\gamma}$$

and its set of weights Ω_3 must be stable under the action of the Weyl group $W(L_3, T) = \langle s_1, s_2, s_4\rangle$. Our aim is to show that

$$\text{either} \quad \Omega_3 = \Gamma_3 \cap \Phi_< \quad \text{or} \quad \Omega_3 = \Gamma_3. \tag{3.1.12}$$

First, let us consider the partition of Γ_3 as disjoint union of the following subsets :

$$\begin{aligned}
\Lambda_1 :=&\{\alpha_1 + 2\alpha_2 + 4\alpha_3 + 2\alpha_4,\ \alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4,\ \widetilde{\alpha}\},\\
\Lambda_2 :=&\{\alpha_1 + 2\alpha_2 + 2\alpha_3,\ \alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4,\ \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4,\ \alpha_2 + 2\alpha_3 + 2\alpha_4,\\
&\quad \alpha_2 + 2\alpha_3, \alpha_1 + \alpha_2 + 2\alpha_3\},\\
\Lambda_3 :=&\{\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4,\ \alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4\},\\
\Lambda_4 :=&\{\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4,\ \alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4,\ \alpha_2 + 2\alpha_3 + \alpha_4\},\\
\Lambda_5 :=&\{\alpha_2 + \alpha_3 + \alpha_4,\ \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4,\ \alpha_1 + \alpha_2 + \alpha_3,\ \alpha_2 + \alpha_3,\ \alpha_3,\ \alpha_3 + \alpha_4\}.
\end{aligned}$$

Notice that $\Lambda_1 \cup \Lambda_2 = \Gamma_3 \cap \Phi_>$ and $\Lambda_3 \cup \Lambda_4 \cup \Lambda_5 = \Gamma_3 \cap \Phi_<$; moreover, as in the preceding case, let us remark that Λ_1, Λ_3, $\Lambda_2 \cup \Lambda_4$ and Λ_5 are stable under $W(L_3, T)$ because their elements have as coefficient respectively 4, 3, 2 and 1 with respect to the simple root α_3. Now let us prove by direct computation that :

- Λ_1 is stable by $W(L_3, T)$:

$$\begin{aligned}
s_2(\alpha_1 + 2\alpha_2 + 4\alpha_3 + 2\alpha_4) &= \alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4,\\
s_1(\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4) &= \widetilde{\alpha};
\end{aligned}$$

- Λ_2 is stable by $W(L_3, T)$:

$$\begin{aligned}
s_4(\alpha_1 + 2\alpha_2 + 2\alpha_3) &= \alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4,\\
s_2(\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4) &= \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4,\\
s_1(\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4) &= \alpha_2 + 2\alpha_3 + 2\alpha_4,\\
s_4(\alpha_2 + 2\alpha_3 + 2\alpha_4) &= \alpha_2 + 2\alpha_3,\\
s_1(\alpha_2 + 2\alpha_3) &= \alpha_1 + \alpha_2 + 2\alpha_3;
\end{aligned}$$

- Λ_3 is stable by $W(L_3, T)$:

$$s_4(\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4) = \alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4;$$

- Λ_4 is stable by $W(L_3, T)$:

$$\begin{aligned}
s_2(\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4) &= \alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4,\\
s_1(\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4) &= \alpha_2 + 2\alpha_3 + \alpha_4;
\end{aligned}$$

- Λ_5 is stable by $W(L_3, T)$:

$$s_1(\alpha_2+\alpha_3+\alpha_4) = \alpha_1+\alpha_2+\alpha_3+\alpha_4,$$
$$s_4(\alpha_1+\alpha_2+\alpha_3+\alpha_4) = \alpha_1+\alpha_2+\alpha_3,$$
$$s_1(\alpha_1+\alpha_2+\alpha_3) = \alpha_2+\alpha_3,$$
$$s_2(\alpha_2+\alpha_3) = \alpha_3,$$
$$s_4(\alpha_3) = \alpha_3+\alpha_4.$$

Thus, for $j = 1, \dots, 5$, we have shown that $\Lambda_j \cap \Omega_3 \neq \emptyset$ implies that $\Lambda_j \subset \Omega_3$. Next, we need to prove the following claims by using Lemma 3.1.5 on structure constants :

(a) $\Lambda_1 \subset \Omega_3$ implies that $\Lambda_2 \subset \Omega_3$,
(b) $\Lambda_2 \subset \Omega_3$ implies that $\Lambda_1 \subset \Omega_3$,
(c) $\Lambda_3 \subset \Omega_3$ implies that $\Lambda_4 \subset \Omega_3$,
(d) $\Lambda_4 \subset \Omega_3$ implies that $\Lambda_5 \subset \Omega_3$,
(e) $\Lambda_5 \subset \Omega_3$ implies that $\Lambda_4 \subset \Omega_3$,
(f) $\Lambda_4 \cup \Lambda_5 \subset \Omega_3$ implies that $\Lambda_3 \subset \Omega_3$,
(g) $\Lambda_1 \subset \Omega_3$ implies that $\Lambda_3 \subset \Omega_3$.

The parabolic subgroup P being non-reduced by assumption, the set Ω_3 is nonempty hence, once these implications are proved, it must be either all of Γ_3 or $\Lambda_3 \cup \Lambda_4 \cup \Lambda_5 = \Gamma_3 \cap \Phi_<$, which proves (3.1.12).

(a): By assumption $\mathfrak{g}_{-\widetilde{\alpha}} \subset \operatorname{Lie} P$. Set $\gamma = -\widetilde{\alpha}$ and $\delta = \alpha_1 + 2\alpha_2 + 2\alpha_3$, then $\gamma - \delta = -3\alpha_1 + 5\alpha_2 + 6\alpha_3 + 2\alpha_4$ is not a root hence

$$[X_\gamma, X_\delta] = \pm X_{-\alpha_1-\alpha_2-2\alpha_3-2\alpha_4} \in \operatorname{Lie} P$$

so $\alpha_1+\alpha_2+2\alpha_3+2\alpha_4 \in \Lambda_2 \cap \Omega_3$.

(b): By assumption $\mathfrak{g}_{-\alpha_1-2\alpha_2-2\alpha_3} \oplus \mathfrak{g}_{-\alpha_1-\alpha_2-2\alpha_3-2\alpha_4} \subset \operatorname{Lie} P$. Set $\gamma = -\alpha_1 - 2\alpha_2 - 2\alpha_3$ and $\delta = -\alpha_1 - \alpha_2 - 2\alpha_3 - 2\alpha_4$, then $\gamma - \delta = -\alpha_2 + 2\alpha_4$ is not a root hence

$$[X_\gamma, X_\delta] = \pm X_{-\widetilde{\alpha}} \in \operatorname{Lie} P$$

so $\widetilde{\alpha} \in \Lambda_1 \cap \Omega_3$.

(c): By assumption $\mathfrak{g}_{-\alpha_1-2\alpha_2-3\alpha_3-2\alpha_4} \subset \operatorname{Lie} P$. Set $\gamma = -\alpha_1 - 2\alpha_2 - 3\alpha_3 - 2\alpha_4$ and $\delta = \alpha_3 + \alpha_4 \in \Phi^+$, then $\gamma - \delta = -\alpha_1 - 2\alpha_2 - 4\alpha_3 - 3\alpha_4$ is not a root hence

$$[X_\gamma, X_\delta] = \pm X_{-\alpha_1-2\alpha_2-2\alpha_3-\alpha_4} \in \operatorname{Lie} P$$

so $\alpha_1+2\alpha_2+2\alpha_3+\alpha_4 \in \Lambda_4 \cap \Omega_3$.

(d): By assumption $\mathfrak{g}_{-\alpha_2-2\alpha_3-\alpha_4} \subset \operatorname{Lie} P$. Set $\gamma = -\alpha_2 - 2\alpha_3 - \alpha_4$ and $\delta = \alpha_3 \in \Phi^+$, then $\gamma - \delta = -\alpha_2 - 3\alpha_3 - \alpha_4$ is not a root hence

$$[X_\gamma, X_\delta] = \pm X_{-\alpha_2-\alpha_3-\alpha_4} \in \operatorname{Lie} P$$

so $\alpha_2+\alpha_3+\alpha_4 \in \Lambda_5 \cap \Omega_3$.

(e): By assumption $\mathfrak{g}_{-\alpha_3-\alpha_4} \oplus \mathfrak{g}_{-\alpha_2-\alpha_3} \subset \operatorname{Lie} P$. Set $\gamma = -\alpha_3 - \alpha_4$ and $\delta = -\alpha_2 - \alpha_3$, then $\gamma - \delta = -\alpha_2 + \alpha_4$ is not a root hence

$$[X_\gamma, X_\delta] = \pm X_{-\alpha_2-2\alpha_3-\alpha_4} \in \operatorname{Lie} P$$

so $\alpha_2+2\alpha_3+\alpha_4 \in \Lambda_4 \cap \Omega_3$.

(f): By assumption $\mathfrak{g}_{-\alpha_1-\alpha_2-\alpha_3-\alpha_4} \oplus \mathfrak{g}_{-\alpha_2-2\alpha_3-\alpha_4} \subset \operatorname{Lie} P$. Set $\gamma = -\alpha_1 - \alpha_2 - \alpha_3 - \alpha_4$ and $\delta = -\alpha_2 - 2\alpha_3 - \alpha_4$, then $\gamma - \delta = -\alpha_1 - \alpha_3$ is not a root hence

$$[X_\gamma, X_\delta] = \pm X_{-\alpha_1-2\alpha_2-3\alpha_3-2\alpha_4} \in \operatorname{Lie} P$$

so $\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 \in \Lambda_3 \cap \Omega_3$.

(g): By assumption $\mathfrak{g}_{-\alpha_1-2\alpha_2-4\alpha_3-2\alpha_4} \subset \operatorname{Lie} P$. Set $\gamma = -\alpha_1 - 2\alpha_2 - 4\alpha_3 - 2\alpha_4$ and $\delta = \alpha_3 \in \Phi^+$, then $\gamma - \delta = -\alpha_1 - 2\alpha_2 - 5\alpha_3 - 2\alpha_4$ is not a root hence

$$[X_\gamma, X_\delta] = \pm X_{-\alpha_1-2\alpha_2-3\alpha_3-2\alpha_4} \in \operatorname{Lie} P$$

so $\alpha_1 - 2\alpha_2 + 3\alpha_3 + 2\alpha_4 \in \Lambda_3 \cap \Omega_3$.

Case $P_{\mathbf{red}} = P_4$.

Let us assume that $P_{\mathrm{red}} = P_4$ and fix the analogous notation as above: the Levi subgroup $L_4 := P_4 \cap P_4^-$, which is of type B_3, acts on

$$W_4 := \operatorname{Lie} P / \operatorname{Lie} P_4 = \bigoplus_{\gamma\in\Omega_4} \mathfrak{g}_{-\gamma} \subset V_4 := \operatorname{Lie} G / \operatorname{Lie} P_4 = \bigoplus_{\gamma\in\Gamma_4} \mathfrak{g}_{-\gamma}$$

and its set of weights Ω_4 must be stable under the action of the Weyl group $W(L_4, T) = \langle s_1, s_2, s_3 \rangle$. Our aim is to show that

$$\text{either} \quad \Omega_4 = \Gamma_4 \cap \Phi_< \quad \text{or} \quad \Omega_4 = \Gamma_4. \tag{3.1.13}$$

Let $\beta := \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4$ and consider, as in the first case of this proof, the action of $W(L_4, T)$ on

$$(\Gamma_4 \cap \Phi_<) \backslash \{\beta, \alpha_4\} := \{\alpha_3 + \alpha_4,\ \alpha_2 + \alpha_3 + \alpha_4,\ \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4,\ \alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4,$$
$$\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4,\ \alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4,\ \alpha_2 + 2\alpha_3 + \alpha_4\},$$

which is transitive because

$$\begin{aligned}
&s_2(\alpha_3 + \alpha_4) = \alpha_2 + \alpha_3 + \alpha_4\\
&s_1(\alpha_2 + \alpha_3 + \alpha_4) = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4,\\
&s_3(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) = \alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4,\\
&s_2(\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4) = \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4,\\
&s_1(\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4) = \alpha_2 + 2\alpha_3 + \alpha_4,\\
&s_3(\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4) = \alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4,
\end{aligned}$$

and the same action on

$$\Gamma_4 \cap \Phi_> = \{\alpha_2 + 2\alpha_3 + 2\alpha_4,\ \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4,\ \alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4,$$
$$\alpha_1 + 2\alpha_2 + 4\alpha_3 + 2\alpha_4,\ \alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4,\ \widetilde{\alpha}\},$$

which is also transitive because

$$\begin{aligned} s_1(\alpha_2+2\alpha_3+2\alpha_4) &= \alpha_1+\alpha_2+2\alpha_3+2\alpha_4, \\ s_2(\alpha_1+\alpha_2+2\alpha_3+2\alpha_4) &= \alpha_1+2\alpha_2+2\alpha_3+2\alpha_4, \\ s_3(\alpha_1+2\alpha_2+2\alpha_3+2\alpha_4) &= \alpha_1+2\alpha_2+4\alpha_3+2\alpha_4, \\ s_2(\alpha_1+2\alpha_2+4\alpha_3+2\alpha_4) &= \alpha_1+3\alpha_2+4\alpha_3+2\alpha_4, \\ s_1(\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4) &= \widetilde{\alpha}. \end{aligned}$$

Next, we prove the following claims using Lemma 3.1.5 on structure constants :

(a) $\Gamma_4 \cap \Phi_> \subset \Omega_4$ implies that $\beta \in \Omega_4$,
(b) $\beta \in \Omega_4$ implies that $(\Gamma_4 \cap \Phi_<)\backslash\{\beta, \alpha_4\} \subset \Omega_4$,
(c) $(\Gamma_4 \cap \Phi_<)\backslash\{\beta, \alpha_4\} \subset \Omega_4$ implies that $\alpha_4 \in \Omega_4$,
(d) $\alpha_4 \in \Omega_4$ implies that $(\Gamma_4 \cap \Phi_<)\backslash\{\beta, \alpha_4\} \subset \Omega_4$,
(e) $(\Gamma_4 \cap \Phi_<)\backslash\{\beta\} \subset \Omega_4$ implies that $\beta \subset \Omega_4$.

The parabolic subgroup P being non-reduced by assumption, the set Ω_4 is nonempty hence, once these implications are proved, it must be either all of Γ_4 or $\Gamma_4 \cap \Phi_<$, which proves (3.1.13).

(a): By assumption $\mathfrak{g}_{-\alpha_2-2\alpha_3-2\alpha_4} \subset \operatorname{Lie} P$ and $\mathfrak{g}_{-\alpha_1-\alpha_2-\alpha_3} \in \operatorname{Lie} L_4 \subset \operatorname{Lie} P$. Set $\gamma = -\alpha_2 - 2\alpha_3 - 2\alpha_4$ and $\delta = -\alpha_1 - \alpha_2 - \alpha_3$, then $\gamma - \delta = \alpha_1 - \alpha_3 - 2\alpha_4$ is not a root hence

$$[X_\gamma, X_\delta] = \pm X_{-\beta} \in \operatorname{Lie} P$$

so $\beta \in \Omega_4$.

(b): By assumption $\mathfrak{g}_{-\beta} \subset \operatorname{Lie} P$. Set $\gamma = -\beta$ and $\delta = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \in \Phi^+$, then $\gamma - \delta = -2\alpha_1 - 3\alpha_2 - 4\alpha_3 - 3\alpha_4$ is not a root hence

$$[X_\gamma, X_\delta] = \pm X_{-\alpha_2-2\alpha_3-\alpha_4} \in \operatorname{Lie} P$$

so $\alpha_2 + 2\alpha_3 + \alpha_4 \in ((\Gamma_4 \cap \Phi_<)\backslash\{\beta, \alpha_4\}) \cap \Omega_4$.

(c): By assumption $\mathfrak{g}_{-\alpha_3-\alpha_4} \subset \operatorname{Lie} P$. Set $\gamma = -\alpha_3 - \alpha_4$ and $\delta = \alpha_3 \in \Phi^+$, then $\gamma - \delta = -2\alpha_3 - \alpha_4$ is not a root hence

$$[X_\gamma, X_\delta] = \pm X_{-\alpha_4} \in \operatorname{Lie} P$$

so $\alpha_4 \in \Omega_4$.

(d): By assumption $\mathfrak{g}_{-\alpha_4} \subset \operatorname{Lie} P$ and $\mathfrak{g}_{-\alpha_3} \subset \operatorname{Lie} L_4 \subset \operatorname{Lie} P$. Set $\gamma = -\alpha_4$ and $\delta = -\alpha_3$, then $\gamma - \delta = \alpha_3 - \alpha_4$ is not a root hence

$$[X_\gamma, X_\delta] = \pm X_{-\alpha_3-\alpha_4} \in \operatorname{Lie} P$$

so $\alpha_3 + \alpha_4 \in ((\Gamma_4 \cap \Phi_<)\backslash\{\beta, \alpha_4\}) \cap \Omega_4$.

(e): By assumption $\mathfrak{g}_{-\alpha_4} \oplus \mathfrak{g}_{-\alpha_1-2\alpha_2-3\alpha_3-\alpha_4} \subset \operatorname{Lie} P$. Set $\gamma = -\alpha_4$ and $\delta = -\alpha_1 - 2\alpha_2 - 3\alpha_3 - \alpha_4$, then $\gamma - \delta = \alpha_1 + 2\alpha_2 + 3\alpha_3$ is not a root hence

$$[X_\gamma, X_\delta] = \pm X_{-\beta} \in \operatorname{Lie} P$$

so $\beta \in \Omega_4$.

Conclusion: up to this point all computations hold in both characteristic $p = 2$ and 3. To conclude our proof when $p = 3$, one more step - which works simultaneously for all cases $i = 1, 2, 3, 4$ - is necessary in order to conclude that $\Omega_i = \Gamma_i$. That is, we want to show that $(\Gamma_i \cap \Phi_<) \subset \Omega_i$ implies $(\Gamma_i \cap \Phi_>) \cap \Omega_i \neq \emptyset$. By assumption, $\mathfrak{g}_{-\alpha_1-2\alpha_2-3\alpha_3-2\alpha_4} \subset \operatorname{Lie} P$. Set $\gamma = -\alpha_1 - 2\alpha_2 - 3\alpha_3 - 2\alpha_4$ and $\delta = \alpha_3 \in \Phi^+$, then $\gamma + \delta$ and $\gamma - \delta$ are still roots while $\gamma - 2\delta = -\alpha_1 - 2\alpha_2 - 5\alpha_3 - 2\alpha_4$ is not, hence

$$[X_\gamma, X_\delta] = \pm 2 X_{-\alpha_1-2\alpha_2-4\alpha_3-2\alpha_4} \in \operatorname{Lie} P \quad \text{hence } X_{-\alpha_1-2\alpha_2-4\alpha_3-2\alpha_4} \in \operatorname{Lie} P,$$

so that $-\alpha_1 - 2\alpha_2 - 4\alpha_3 - 2\alpha_4 \in (\Gamma_i \cap \Phi_>) \cap \Omega_i$ as wanted. □

Proof. (**of Theorem 3.1.2 in type F_4**)
Let G be simple of type F_4 and $X = G/P$ with a faithful G-action such that P_{red} is maximal and P is nonreduced. When $p = 2$, Proposition 3.1.16 implies that $\mathfrak{g}_< \subset \operatorname{Lie} P$, hence we get $\operatorname{Lie} N_G \subset \operatorname{Lie} P$ and hence $N_G \subset P$ by the equivalence of categories, which is a contradiction by Remark 3.1.4. When $p = 3$, the above Proposition implies that $\operatorname{Lie} P = \operatorname{Lie} G$, hence the Frobenius kernel satisfies $G_1 \subset P$, which gives again a contradiction. Therefore in both cases P must be a reduced parabolic. □

3.2. The case of type G_2

The last non-simply laced Dynkin diagram we have to consider is of a group G of type G_2. In this case, things behave as expected when the reduced parabolic subgroup is P^{α_2}, the one associated with the long simple root α_2, or when the characteristic is $p = 3$: the proof follows the same strategy as in types B_n, C_n and F_4.

This still leaves out the case of a nonreduced parabolic subgroup satisfying $P_{\mathrm{red}} = P^{\alpha_1}$ in characteristic 2, where α_1 denotes the short simple root. Under such assumptions, we find two maximal p-Lie subalgebras

$$\mathfrak{h} := \operatorname{Lie} P^{\alpha_1} \oplus \mathfrak{g}_{-2\alpha_1-\alpha_2} \quad \text{and} \quad \mathfrak{l} := \operatorname{Lie} P^{\alpha_1} \oplus \mathfrak{g}_{-\alpha_1} \oplus \mathfrak{g}_{-\alpha_1-\alpha_2},$$

containing $\operatorname{Lie} P^{\alpha_1}$. Let H and L be the subgroups of G of height one with Lie algebra respectively equal to $\mathfrak{g}_{-2\alpha_1-\alpha_2}$ and $\mathfrak{g}_{-\alpha_1} \oplus \mathfrak{g}_{-\alpha_1-\alpha_2}$, and define

$$P_{\mathfrak{h}} := \langle H, P^{\alpha_1} \rangle \quad \text{and} \quad P_{\mathfrak{l}} := \langle L, P^{\alpha_1} \rangle.$$

This gives rise to two parabolic subgroups which have as reduced subgroup a maximal one, but cannot be described as $(\ker \varphi) P^{\alpha_1}$ for some isogeny φ with source G. We then move on to investigate the corresponding homogeneous spaces, which we describe by using the description of G as automorphism group of an octonion algebra.
The main result is the following, which completes the classification of Theorem 3.1.1.

Theorem 3.2.1. *Let G be of type G_2 in characteristic two and let P be a nonreduced parabolic subgroup of G having P^{α_1} as reduced part.*
Then one of the three following cases holds:

- *P is of standard type and $X \simeq G/P^{\alpha_1}$ is isomorphic to a quadric Q in $\mathbf{P}^6$;*
- *P is obtained from $P_{\mathfrak{h}}$ by pull back via an iterated Frobenius morphism and $X \simeq G/P_{\mathfrak{h}}$ is isomorphic to $\mathbf{P}^5$;*
- *P is obtained from $P_{\mathfrak{l}}$ by pull back via an iterated Frobenius morphism and $X \simeq G/P_{\mathfrak{l}}$ is isomorphic to a hyperplane section of $\operatorname{Sp}_6 / P^{\alpha_3}$.*

Remark 3.2.2. In a later stage of the writing of this manuscript, the following useful reference was pointed out to me: [**Hog**] classifies the restricted Lie subalgebras of $\operatorname{Lie} G$, where G is a group of type G_2 in characteristic $p = 2$.

Let us recall for reference the following result: see [**Dem**, Theorem 1], reformulated here under the stronger hypothesis of k being an algebraically closed field. Concerning the notion of automorphism group, see Section 2.3 above.

THEOREM 3.2.3. *Let H' be a semisimple adjoint group over k and Q' a reduced parabolic subgroup of H'. Then the natural homomorphism*

$$H' \longrightarrow H := \underline{\mathrm{Aut}}^0_{H'/Q'}$$

is an isomorphism in all but the three following cases:

(a) H' is of type C_n for some $n \geq 2$ and $Q' = P^{\alpha_1}$ is associated to the first short simple root: in this case the automorphism group H is simple adjoint of type A_{2n-1};

(b) H' is of type B_n for some $n \geq 2$ and $Q' = P^{\alpha_n}$ is associated to the short simple root: in this case the automorphism group H is simple adjoint of type D_{n+1};

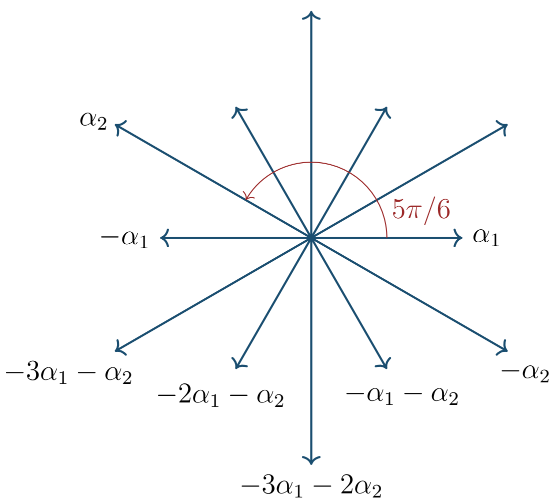
(c) H' is of type G_2 and $Q' = P^{\alpha_1}$: in this case the automorphism group H is simple adjoint of type B_3.

Let Q be the reduced parabolic subgroup of H such that $H := \underline{\mathrm{Aut}}^0_{H/Q}$. With a slight change of notation compared to Demazure, we call the three pairs (H, Q) in the cases (a), (b) and (c) of the Theorem *exceptional*, while (H', Q') is called the *associated* pair to the exceptional one. The result above is needed to conclude the type G_2 case, as well as later on, when dealing with higher Picard ranks.

3.2.1. What works as expected. Let us consider a group G with root system of type G_2 over a field k of characteristic $p = 2$ or 3. Following notations from [**Bou**], the elements of Φ^+ are

$$\alpha_1, \quad \alpha_1 + \alpha_2, \quad 2\alpha_1 + \alpha_2, \quad 3\alpha_1 + \alpha_2, \quad \alpha_2, \quad 3\alpha_1 + 2\alpha_2.$$

In particular, let us consider as elements of the basis Δ the short root α_1 and the long root α_2; then denote $P_1 := P^{\alpha_1}$ and $P_2 := P^{\alpha_2}$ the associated maximal reduced parabolic subgroups.



Let us recall that, when $p = 3$, $N_G \subset G$ is in this case the unique subgroup of height one such that

$$\operatorname{Lie} N_G = \operatorname{Lie} \alpha_1^\vee(\mathbf{G}_m) \oplus \mathfrak{g}_{\alpha_1} \oplus \mathfrak{g}_{-\alpha_1} \oplus \mathfrak{g}_{\alpha_2+2\alpha_1} \oplus \mathfrak{g}_{-\alpha_2-2\alpha_1} \oplus \mathfrak{g}_{\alpha_1+\alpha_2} \oplus \mathfrak{g}_{-\alpha_1-\alpha_2}.$$

Proposition 3.2.4. *Assume given a nonreduced parabolic P such that $P_{red} = P_1$ (with $p = 3$) or $P_{red} = P_2$ (with $p = 2$ or 3). Then* $\operatorname{Lie} P = \operatorname{Lie} G$ *or* $\operatorname{Lie} P = \operatorname{Lie} P_{red} + \mathfrak{g}_<$. *If $p = 2$, then necessarily* $\operatorname{Lie} P = \operatorname{Lie} G$.

Remark 3.2.5. We can conclude that Theorem 3.1.2 holds in this case as follows: let G be simple of type G_2 and $X = G/P$ with a faithful G-action such that P_{red} is maximal, satisfies the hypothesis of Proposition 3.2.4, and such that P is nonreduced. The above Proposition implies that

$$\operatorname{Lie} \alpha_1^\vee(\mathbf{G}_m) \oplus \mathfrak{g}_< = \operatorname{Lie} N_G \subset \operatorname{Lie} P,$$

hence we get $N_G \subset P$, which is a contradiction by Remark 3.1.4. Therefore P must be a smooth parabolic.

PROOF. **Case $P_{\mathrm{red}} = P_1$.**
Let us assume that $P_{\mathrm{red}} = P_1$ and that the characteristic is $p = 3$. The Levi subgroup $L_1 := P_1 \cap P_1^-$ has root system $\{\pm\alpha_2\}$ and acts on the vector space

$$V_1 := \operatorname{Lie} G / \operatorname{Lie} P_1 = \mathfrak{g}_{-\alpha_1} \oplus \mathfrak{g}_{-\alpha_1-\alpha_2} \oplus \mathfrak{g}_{-2\alpha_1-\alpha_2} \oplus \mathfrak{g}_{-3\alpha_1-\alpha_2} \oplus \mathfrak{g}_{-3\alpha_1-2\alpha_2}.$$

Now, let us look at the nonzero vector subspace $W_1 := \operatorname{Lie} P / \operatorname{Lie} P_1$, which is in particular an L_1-submodule of V_1. Thus, the set of its weights must be stable under the reflection s_{α_2}. This means, by a direct computation, that

$$\mathfrak{g}_{-3\alpha_1-2\alpha_2} \subset W_1 \iff \mathfrak{g}_{-3\alpha_1-\alpha_2} \subset W_1, \tag{3.2.1}$$

$$\mathfrak{g}_{-\alpha_1-\alpha_2} \subset W_1 \iff \mathfrak{g}_{-\alpha_1} \subset W_1. \tag{3.2.2}$$

Let us assume first that $\mathfrak{g}_{-\alpha_1-\alpha_2} \oplus \mathfrak{g}_{-\alpha_1} \subset W_1$. Then, applying Lemma 3.1.5 to $\gamma = -\alpha_1 - \alpha_2$ and $\delta = -\alpha_1$ gives

$$[X_{-\alpha_1-\alpha_2}, X_{-\alpha_1}] = \pm 2X_{-2\alpha_1-\alpha_2}, \quad \text{hence } X_{-2\alpha_1-\alpha_2} \in \operatorname{Lie} P,$$

since $\gamma + \delta$ and $\gamma - \delta$ are roots while $\gamma - 2\delta = \alpha_1 - \alpha_2$ is not. Conversely, assuming $\mathfrak{g}_{-2\alpha_1-\alpha_2} \subset W_1$ and considering roots $\gamma = -2\alpha_1 - \alpha_2$ and $\delta = \alpha_1$ yields

$$[X_{-2\alpha_1-\alpha_2}, X_{\alpha_1}] = \pm 2X_{-\alpha_1-\alpha_2}, \quad \text{hence } X_{-\alpha_1-\alpha_2} \in \operatorname{Lie} P.$$

In other words, we have showed that whenever a root subspace associated to a short negative root is contained in W_1, the other two are too.
To conclude this first case, it is enough to show that

$$\mathfrak{g}_{-3\alpha_1-2\alpha_2} \oplus \mathfrak{g}_{-3\alpha_1-\alpha_2} \subset W_1 \quad \text{implies that} \quad \mathfrak{g}_{-2\alpha_1-\alpha_2} \subset W_1.$$

This can be done by considering roots $\gamma = -3\alpha_1 - 2\alpha_2$ and $\delta = \alpha_1 + \alpha_2$, for which $\gamma + \delta$ is a root but $\gamma - \delta = -4\alpha_1 - 3\alpha_2$ is not, hence

$$[X_{-3\alpha_1-2\alpha_2}, X_{\alpha_1+\alpha_2}] = \pm X_{-2\alpha_1-\alpha_2} \in \operatorname{Lie} P.$$

Case $P_{\mathbf{red}} = P_2$.

Moving on to the second case, let us assume that $P_{\mathrm{red}} = P_2$. The Levi subgroup $L_2 := P_2 \cap P_2^-$ has root system $\{\pm\alpha_1\}$ and acts on the vector space

$$V_2 := \operatorname{Lie} G/\operatorname{Lie} P_1 = \mathfrak{g}_{-\alpha_2} \oplus \mathfrak{g}_{-\alpha_1-\alpha_2} \oplus \mathfrak{g}_{-2\alpha_1-\alpha_2} \oplus \mathfrak{g}_{-3\alpha_1-\alpha_2} \oplus \mathfrak{g}_{-3\alpha_1-2\alpha_2}.$$

Now, let us look at the nonzero vector subspace $W_2 := \operatorname{Lie} P/\operatorname{Lie} P_2$, which is in particular an L_2-submodule of V_2. Thus, the set of its weights must be stable under the reflection s_{α_1}. This means, by a direct computation, that

$$\mathfrak{g}_{-\alpha_1-\alpha_2} \subset W_2 \iff \mathfrak{g}_{-2\alpha_1-\alpha_2} \subset W_2, \tag{3.2.3}$$

$$\mathfrak{g}_{-3\alpha_1-\alpha_2} \subset W_2 \iff \mathfrak{g}_{-\alpha_1} \subset W_2. \tag{3.2.4}$$

The equivalence (3.2.3) already implies that once a root subspace associated to a short negative root is contained in W_2, the only other one is too.

If $p = 3$, to conclude it suffices to show that $\mathfrak{g}_{-\gamma} \subset W_2$ for some long root $\gamma \in \Phi^+$ implies $W_2 = V_2$ i.e. $\operatorname{Lie} P = \operatorname{Lie} G$. First,

$$[X_{-3\alpha_1-2\alpha_2}, X_{\alpha_2}] = \pm X_{-3\alpha_1-\alpha_2},$$

because $(-3\alpha_1 - 2\alpha_2) - \alpha_2$ is not a root, and conversely

$$[X_{-3\alpha_1-\alpha_2}, X_{-\alpha_2}] = \pm X_{-3\alpha_1-2\alpha_2},$$

because $(-3\alpha_1 - \alpha_2) - (-\alpha_2)$ is not a root. Finally,

$$[X_{-3\alpha_1-\alpha_2}, X_{\alpha_1}] = \pm X_{-2\alpha_1-\alpha_2},$$

because $(-3\alpha_1 - \alpha_2) - \alpha_1$ is not a root. This proves that in this case $W_2 = V_2$.

If $p = 2$, one more step must be added: assume that $\mathfrak{g}_{-2\alpha_1-\alpha_2} \oplus \mathfrak{g}_{-\alpha_1-\alpha_2} \subset W_2$, then

$$[X_{-2\alpha_1-\alpha_2}, X_{-\alpha_1}] = \pm X_{-3\alpha_1-\alpha_2}, \quad \text{hence } X_{-3\alpha_1-\alpha_2} \in \operatorname{Lie} P,$$

because $(-2\alpha_1-\alpha_2)+\alpha_1$, $(-2\alpha_1-\alpha_2)+2\alpha_1$ are roots, while $(-2\alpha_1-\alpha_2)+3\alpha_1$ is not. This last remark, together with the above computations shows that when $p = 2$ necessarily $\operatorname{Lie} P = \operatorname{Lie} G$. □

3.2.2. What does not. The only case yet to consider is the following: the characteristic is $p = 2$, the group G is of type G_2 and P is a nonreduced parabolic subgroup satisfying $P_{\mathrm{red}} = P^{\alpha_1}$, the reduced parabolic associated to the short simple root, whose Levi subgroup has root system $\{\pm\alpha_2\}$. Let us place ourselves in this setting: by repeating the same reasoning as above, we can obtain only a weaker statement.

Lemma 3.2.6. *Assume that one of the two root subspaces associated to $-3\alpha_1 - 2\alpha_2$ and $-3\alpha_1 - \alpha_2$ is contained in* $\operatorname{Lie} P$*. Then* $\operatorname{Lie} P = \operatorname{Lie} G$.

PROOF. By (3.2.1), we have that both root subspaces are in $\operatorname{Lie} P$. Then considering roots $\gamma = -3\alpha_1 - 2\alpha_2$, $\delta = \alpha_1 + \alpha_2$ and $\delta' = 2\alpha_1 + \alpha_2$ yields

$$[X_\gamma, X_\delta] = \pm X_{-2\alpha_1-\alpha_2} \in \operatorname{Lie} P \quad \text{and} \quad [X_\gamma, X_{\delta'}] = \pm X_{-\alpha_1-\alpha_2} \in \operatorname{Lie} P,$$

because $\gamma - \delta$ and $\gamma - \delta'$ are not roots. This means that if one long root is added then we have to add everything else. □

The same reasoning applied to short roots fails, due to the vanishing of structure constants in characteristic 2. More precisely, we can identify two Lie subalgebras strictly containing $\operatorname{Lie} P^{\alpha_1}$, which cannot be Lie ideals since $\operatorname{Lie} G$ is a simple p-Lie algebra (see Lemma 2.5.9) as follows: define the following vector subspaces

(3.2.5) $$\mathfrak{h} := \operatorname{Lie} P^{\alpha_1} \oplus \mathfrak{g}_{-2\alpha_1-\alpha_2} = \operatorname{Lie} B \oplus \mathfrak{g}_{-\alpha_2} \oplus \mathfrak{g}_{-2\alpha_1-\alpha_2};$$

(3.2.6) $$\mathfrak{l} := \operatorname{Lie} P^{\alpha_1} \oplus \mathfrak{g}_{-\alpha_1} \oplus \mathfrak{g}_{-\alpha_1-\alpha_2} = \operatorname{Lie} B \oplus \mathfrak{g}_{-\alpha_2} \oplus \mathfrak{g}_{-\alpha_1} \oplus \mathfrak{g}_{-\alpha_1-\alpha_2}.$$

Lemma 3.2.7. *With the above notation, $\mathfrak{h}$ and $\mathfrak{l}$ are two p-Lie subalgebras of* $\operatorname{Lie} G$.

PROOF. Let $\{X_\gamma : \gamma \in \Phi, H_{\alpha_1}, H_{\alpha_2}\}$ be a Chevalley basis of $\operatorname{Lie} G$. First, using Lemma 3.1.5 we can calculate a few structure constants which are then useful in the rest of the proof:

	$\operatorname{ad}(X_{-2\alpha_1-\alpha_2})$	$\operatorname{ad}(X_{-\alpha_1})$	$\operatorname{ad}(X_{-\alpha_1-\alpha_2})$	$\operatorname{ad}(X_{\alpha_1})$	$\operatorname{ad}(X_{2\alpha_1+\alpha_2})$
X_{α_1}	0	$\in \operatorname{Lie} T$	$X_{-\alpha_2}$	0	$X_{3\alpha_1+\alpha_2}$
$X_{3\alpha_1+\alpha_2}$	X_{α_1}	$X_{2\alpha_1+\alpha_2}$	0	0	0
$X_{2\alpha_1+\alpha_2}$	$\in \operatorname{Lie} T$	0	0	$X_{3\alpha_1+\alpha_2}$	0
$X_{3\alpha_1+2\alpha_2}$	$X_{\alpha_1+\alpha_2}$	0	$X_{2\alpha_1+\alpha_2}$	0	0
$X_{\alpha_1+\alpha_2}$	0	X_{α_2}	$\in \operatorname{Lie} T$	0	$X_{3\alpha_1+2\alpha_2}$
X_{α_2}	0	0	$X_{-\alpha_1}$	$X_{\alpha_1+\alpha_2}$	0
$X_{-\alpha_1}$	$X_{-3\alpha_1-\alpha_2}$	0	0	$\in \operatorname{Lie} T$	0
$X_{-3\alpha_1-\alpha_2}$	0	0	0	$X_{-2\alpha_1-\alpha_2}$	$X_{-\alpha_1}$
$X_{-2\alpha_1-\alpha_2}$	0	$X_{-3\alpha_1-\alpha_2}$	$X_{-3\alpha_1-2\alpha_2}$	0	$\in \operatorname{Lie} T$
$X_{-3\alpha_1-2\alpha_2}$	0	0	0	0	$X_{-\alpha_1-\alpha_2}$
$X_{-\alpha_1-\alpha_2}$	$X_{-3\alpha_1-2\alpha_2}$	0	0	$X_{-\alpha_2}$	0
$X_{-\alpha_2}$	0	$X_{-\alpha_1-\alpha_2}$	0	0	0

Let us verify that $\mathfrak{h}$ is a Lie subalgebra. Since we know that $\operatorname{Lie} P^{\alpha_1}$ is one, it is enough to show that $[\mathfrak{g}_{-2\alpha_1-\alpha_2}, \operatorname{Lie} P^{\alpha_1}] \subset \mathfrak{h}$. Lemma 3.1.5 implies that

$$[\mathfrak{g}_{-2\alpha_1-\alpha_2}, \operatorname{Lie} T] = [X_{-2\alpha_1-\alpha_2}, \operatorname{Lie} T] \subset \mathfrak{g}_{-2\alpha_1-\alpha_2} \subset \mathfrak{h}.$$

Moreover, the first column of the above table shows that

$$[\mathfrak{g}_{-2\alpha_1-\alpha_2}, \mathfrak{g}_\gamma] = k[X_{-2\alpha_1-\alpha_2}, X_\gamma] \subset \mathfrak{h},$$

for all roots γ whose root subspace is contained in $\operatorname{Lie} P^{\alpha_1}$.
Analogously, let us prove that $\mathfrak{l}$ is a Lie subalgebra: for this, it is enough to show that

$$[\mathfrak{g}_{-\alpha_1}, \operatorname{Lie} P^{\alpha_1}], [\mathfrak{g}_{-\alpha_1-\alpha_2}, \operatorname{Lie} P^{\alpha_1}], [\mathfrak{g}_{-\alpha_1}, \mathfrak{g}_{-\alpha_1-\alpha_2}] \subset \mathfrak{l}.$$

First, Lemma 3.1.5 implies that

$$[\mathfrak{g}_{-\alpha_1}, \operatorname{Lie} T] = [X_{-\alpha_1}, \operatorname{Lie} T] \subset g_{-\alpha_1} \subset \mathfrak{l};$$
$$[\mathfrak{g}_{-\alpha_1-\alpha_2}, \operatorname{Lie} T] = [X_{-\alpha_1-\alpha_2}, \operatorname{Lie} T] \subset g_{-\alpha_1-\alpha_2} \subset \mathfrak{l}.$$

Moreover, the second and third column in the above table show that

$$[\mathfrak{g}_{-\alpha_1}, \mathfrak{g}_\gamma] = k[X_{-\alpha_1}, X_\gamma] \quad \text{and} \quad [\mathfrak{g}_{-\alpha_1-\alpha_2}, \mathfrak{g}_\gamma] = k[X_{-\alpha_1-\alpha_2}, X_\gamma]$$

are both subspaces of $\mathfrak{l}$, for all roots γ whose root subspace is contained in $\operatorname{Lie} P^{\alpha_1}$.
To conclude there is still left to show that $\mathfrak{h}$ and $\mathfrak{l}$ are stable by the p-mapping (recall that by assumption $p = 2$), knowing that $\operatorname{Lie} P^{\alpha_1}$ is. In other words, setting Y_γ equal to the image of X_γ by the p-mapping, we want to prove that $Y_{-2\alpha_1-\alpha_2} \in \mathfrak{h}$ and that $Y_{-\alpha_1}, Y_{-\alpha_1-\alpha_2} \in \mathfrak{l}$.
To do this, let

$$Y_{-2\alpha_1-\alpha_2} = H + \sum_{\delta\in\Phi} a_\delta X_\delta, \quad \text{for some } a_\delta \in k,\ H \in \operatorname{Lie} T.$$

It is enough to show that $a_{-\alpha_1} = a_{-3\alpha_1-\alpha_2} = a_{-3\alpha_1-2\alpha_2} = a_{-\alpha_1-\alpha_2} = 0$. By the properties of the p-mapping, we have that $\operatorname{ad}(Y_\gamma) = \operatorname{ad}^2(X_\gamma)$ for any root γ. Using that $[X_{-2\alpha_1-\alpha_2}, X_{\alpha_1}]$ vanishes (see table), we have:

$$\begin{aligned} 0 &= \operatorname{ad}(X_{-2\alpha_1-\alpha_2})([X_{-2\alpha_1-\alpha_2}, X_{\alpha_1}]) = \operatorname{ad}^2(X_{-2\alpha_1-\alpha_2})(X_{\alpha_1}) \\ &= \operatorname{ad}(Y_{-2\alpha_1-\alpha_2})(X_{\alpha_1}) = [H, X_{\alpha_1}] + \sum_{\delta\in\Phi} a_\delta [X_\delta, X_{\alpha_1}]. \end{aligned}$$

Expanding all brackets using the fourth column of the above table gives that, for some $a \in k$,

$$0 = aX_{\alpha_1} + a_{2\alpha_1-\alpha_2} X_{3\alpha_1+\alpha_2} + a_{\alpha_2} X_{\alpha_1+\alpha_2} + a_{-\alpha_1} H_{\alpha_1} + a_{-3\alpha_1-\alpha_2} X_{-2\alpha_1-\alpha_2} + a_{-\alpha_1-\alpha_2} X_{-\alpha_2},$$

which implies in particular $a_{-\alpha_1} = a_{-3\alpha_1-\alpha_2} = a_{-\alpha_1-\alpha_2} = 0$. Moreover, $[X_{-2\alpha_1-\alpha_2}, X_{\alpha_1}]$ also vanishes, hence we have

$$\begin{aligned} 0 &= \operatorname{ad}(X_{-2\alpha_1-\alpha_2})([X_{-2\alpha_1-\alpha_2}, X_{\alpha_1+\alpha_2}]) = \operatorname{ad}^2(X_{-2\alpha_1-\alpha_2})(X_{\alpha_1+\alpha_2}) \\ &= \operatorname{ad}(Y_{-2\alpha_1-\alpha_2})(X_{\alpha_1+\alpha_2}) = [H, X_{\alpha_1+\alpha_2}] + \sum_{\delta\in\Phi} a_\delta [X_\delta, X_{\alpha_1+\alpha_2}]. \end{aligned}$$

Writing this with respect to the Chevalley basis gives

$$a_{-3\alpha_1-2\alpha_2}[X_{-3\alpha_1-2\alpha_2}, X_{\alpha_1+\alpha_2}] = a_{-3\alpha_1-2\alpha_2} X_{-2\alpha_1-\alpha_2}$$

as the only term in $X_{-2\alpha_1-\alpha_2}$, meaning that the coefficient $a_{-3\alpha_1-2\alpha_2}$ also vanishes, as wanted: thus we can conclude that $\mathfrak{h}$ is a p-Lie subalgebra of $\operatorname{Lie} G$.
Analogously, let

$$Y_{-\alpha_1} = H' + \sum_{\delta\in\Phi} b_\delta X_\delta, \quad \text{for some } b_\delta \in k, H' \in \operatorname{Lie} T,$$

and as before we aim to show that $b_{-3\alpha_1-\alpha_2} = b_{-2\alpha_1-\alpha_2} = b_{-3\alpha_1-2\alpha_2} = 0$. Using that $[X_{-\alpha_1}, X_{2\alpha_1+\alpha_2}]$ vanishes (see table), we have

$$\begin{aligned} 0 &= \operatorname{ad}(X_{-\alpha_1})([X_{-\alpha_1}, X_{2\alpha_1+\alpha_2}]) = \operatorname{ad}^2(X_{-\alpha_1})([X_{-\alpha_1}, X_{2\alpha_1+\alpha_2}]) \\ &= \operatorname{ad}(Y_{-\alpha_1})(X_{2\alpha_1+\alpha_2}) = [H', X_{2\alpha_1+\alpha_2}] + \sum_{\delta\in\Phi} b_\delta [X_\delta, X_{2\alpha_1+\alpha_2}]. \end{aligned}$$

Expanding all brackets using the last column of the above table gives that, for some $b \in k$ and some $H'' \in \operatorname{Lie} T$,

$$\begin{aligned} 0 = {} & bX_{2\alpha_1+\alpha_2} + b_{\alpha_1} X_{3\alpha_1+\alpha_2} + b_{\alpha_1+\alpha_2} X_{3\alpha_1+2\alpha_2} + b_{-3\alpha_1-\alpha_2} X_{-\alpha_1} + b_{-2\alpha_1-\alpha_2} H'' \\ & + b_{-3\alpha_1-2\alpha_2} X_{-\alpha_1-\alpha_2}. \end{aligned}$$

In particular, this proves that $b_{-3\alpha_1-\alpha_2} = b_{-3\alpha_1-2\alpha_2} = 0$. Moreover, $[X_{-\alpha_1}, X_{-\alpha_1-\alpha_2}]$ also vanishes, hence we have

$$\begin{aligned} 0 &= \mathrm{ad}(X_{-\alpha_1})([X_{-\alpha_1}, X_{-\alpha_1-\alpha_2}]) = \mathrm{ad}^2(X_{-\alpha_1})([X_{-\alpha_1}, X_{-\alpha_1-\alpha_2}]) \\ &= \mathrm{ad}(Y_{-\alpha_1})(X_{-\alpha_1-\alpha_2}) = [H', X_{-\alpha_1-\alpha_2}] + \sum_{\delta\in\Phi} b_\delta [X_\delta, X_{-\alpha_1-\alpha_2}]. \end{aligned}$$

Expanding this with respect to the Chevalley basis gives $b_{-2\alpha_1-\alpha_2}[X_{-2\alpha_1-\alpha_2}, X_{-\alpha_1-\alpha_2}] = b_{-2\alpha_1-\alpha_2} X_{-3\alpha_1-2\alpha_2} X_{-3\alpha_1-2\alpha_2}$ as the only term in $X_{-3\alpha_1-2\alpha_2}$, meaning that the coefficient $b_{-2\alpha_1-\alpha_2}$ also vanishes: this proves that $Y_{-\alpha_1} \in \mathfrak{l}$.
To prove that $Y_{-\alpha_1-\alpha_2}$ is also in $\mathfrak{l}$, an analogous computation, symmetric with respect to the reflection s_{α_2}, can be done. Finally, we can conclude that $\mathfrak{l}$ is a p-Lie subalgebra. □

Corollary 3.2.8. *The p-Lie subalgebras of* $\mathrm{Lie}\, G$ *containing strictly* $\mathrm{Lie}\, P^{\alpha_1}$ *are exactly* $\mathfrak{h}$ *and* $\mathfrak{l}$.

PROOF. Let us consider a p-Lie subalgebra $\mathrm{Lie}\, P^{\alpha_1} \subsetneq \mathfrak{s} \subset \mathrm{Lie}\, G$, meaning that there is some positive root $\gamma \neq \alpha_1$ such that $\mathfrak{g}_{-\gamma}$ is contained in $\mathfrak{s}$. By Lemma 3.2.6, if γ is long then $\mathfrak{s} = \mathrm{Lie}\, G$, so we can assume γ to be short. To do this, let us remark that by Lemma 3.1.5 we have

$$[X_{-\alpha_1}, X_{-2\alpha_1-\alpha_2}] = X_{-3\alpha_1-\alpha_2}, \tag{3.2.7}$$

because $-\alpha_1 - (-2\alpha_1 - \alpha_2))$ and $-\alpha_1 - 2(-2\alpha_1 - \alpha_2))$ are roots while $-\alpha_1 - 3(-2\alpha_1 - \alpha_2))$ is not, hence the structure constant is $3 = 1$. If $\gamma = \alpha_1$, by symmetry with respect to the Weyl group $\{\pm s_{\alpha_2}\}$ of the Levi factor of P^{α_1} we have that $\mathfrak{g}_{-\alpha_1-\alpha_2}$ is also contained in $\mathfrak{s}$, hence either $\mathfrak{s} = \mathfrak{l}$ or it also contains $\mathfrak{g}_{-2\alpha_1-\alpha_2}$. The equality (3.2.7) together with Lemma 3.2.6 then imply $\mathfrak{s} = \mathrm{Lie}\, G$. The same reasoning applies when starting by $\gamma = -\alpha_1 - \alpha_2$. On the other hand, starting by $\gamma = 2\alpha_1 + \alpha_2$ implies that either $\mathfrak{s} = \mathfrak{h}$, or it contains also $\mathfrak{g}_{-\alpha_1} \oplus \mathfrak{g}_{-\alpha_1-\alpha_2}$, from which we conclude again - by (3.2.7) and Lemma 3.2.6 - that $\mathfrak{s} = \mathrm{Lie}\, G$. □

Definition 3.2.9. Let us fix the following notation for the rest of this Section:

(1) $H := (U_{-2\alpha_1-\alpha_2})_1$ is the subgroup of height one such that $\mathrm{Lie}\, H = \mathfrak{g}_{-2\alpha_1-\alpha_2}$, i.e. $\mathfrak{h} = \mathrm{Lie}\, P^{\alpha_1} \oplus \mathrm{Lie}\, H$;
(2) $L := (U_{-\alpha_1})_1 \cdot (U_{-\alpha_1-\alpha_2})_1$ is the subgroup of height one such that $\mathrm{Lie}\, L = \mathfrak{g}_{-\alpha_1} \oplus \mathfrak{g}_{-\alpha_1-\alpha_2}$, i.e. $\mathfrak{l} = \mathrm{Lie}\, P^{\alpha_1} \oplus \mathrm{Lie}\, L$;
(3) $P_{\mathfrak{h}}$ the parabolic subgroup generated by P^{α_1} and H;
(4) $P_{\mathfrak{l}}$ the parabolic subgroup generated by P^{α_1} and L.

Let us notice that $\mathfrak{g}_{-\alpha_1}$ and $\mathfrak{g}_{-\alpha_1-\alpha_2}$ commute, so that L is the direct product of the Frobenius kernels defining it.

Remark 3.2.10. The two parabolic subgroups $P_{\mathfrak{h}}$ and $P_{\mathfrak{l}}$ are *exotic* in the sense that they cannot be of the form $(\ker\varphi)P^{\alpha}$ for some isogeny φ, since when $p = 2$ the only noncentral isogenies in type G_2 are iterated Frobenius homomorphisms (see Proposition 2.5.12).

In the following part we investigate what the homogeneous spaces having as stabilizer respectively $P_{\mathfrak{h}}$ and $P_{\mathfrak{l}}$ are isomorphic to, as varieties.

3.2.2.1. *Parabolic $P_{\mathfrak{h}}$.* Let us denote as Q the smooth quadric in $\mathbf{P}^6$, realized as the homogeneous space

$$\mathrm{SO}_7 / P^{\alpha_1}.$$

Proposition 3.2.11. *Let G be simple of type G_2 in characteristic $p = 2$ and $P_{\mathfrak{h}}$ the parabolic subgroup of Definition 3.2.9. Then the quotient morphism $G/P^{\alpha_1} \longrightarrow G/P_{\mathfrak{h}}$ is the natural projection*

$$\mathbf{P}^6 \supset Q := \{x_3^2 + x_2x_4 + x_1x_5 + x_0x_6 = 0\} \to \mathbf{P}^5, \quad [x_0 : \ldots : x_6] \mapsto [x_0 : x_1 : x_2 : x_4 : x_5 : x_6].$$

In particular, the homogeneous space $G/P_{\mathfrak{h}}$ is isomorphic as a variety to $\mathbf{P}^5 = \mathrm{PSp}_6 / P^{\alpha_1}$.

In order to construct this morphism, we will see the group G as the automorphism group of an octonion algebra - see the Appendix for more details - which is

$$\mathbb{O} = \{(u, v) : u, v \text{ are } 2 \times 2 \text{ matrices}\},$$

with basis $(e_{11}, e_{12}, e_{21}, e_{22}, f_{11}, f_{12}, f_{21}, f_{22})$, recalled in Section 6.1, with unit $e = (1, 0) = e_{11} + e_{22}$, and which is equipped with a norm

$$q(u, v) = \det(u) + \det(v).$$

An embedding of the group G_2 into SO_7 can be seen as follows: let us consider its action on the vector space

$$V := e^{\perp} = \{(u, v) : \det(1 + u) + \det(u) = 1\}$$

as in (6.1.2). This gives an embedding $G_2 \subset \mathrm{SO}_7 = \mathrm{SO}(V)$, which is independent of the characteristic. Since in our setting $p = 2$, we moreover have that have that $e \in V$ hence the group G also acts on the quotient $W := V/ke$, which has dimension 6. This is exactly due to the existence of the very special isogeny from Spin_7 to Sp_6, which factorises through SO_7 and which has been described in Example 2.5.16 in terms of linear algebra. Thus, we get an embedding

$$G_2 \subset \mathrm{Sp}_6 = \mathrm{Sp}(W).$$

By (6.1.4) in the Appendix, a maximal torus T of G - with respect to the basis

$$(f_{12}, f_{11}, e_{12}, e_{21}, f_{22}, f_{21})$$

of W - is given by

$$\mathbf{G}_m{}^2 \ni (a, b) \longmapsto \mathrm{diag}(a, a^{-1}b^{-1}, a^2b, a^{-2}b^{-1}, ab, a^{-1}) = t \in T \subset \mathrm{GL}_6 .$$

Let us recall that the basis of simple roots we fix is $\alpha_1(t) := a$ and $\alpha_2(t) := b$, hence V has the following decomposition in weight spaces :

$$V_0 = ke, V_{\alpha_1} = kf_{12},\ V_{-\alpha_1} = kf_{21},\ V_{\alpha_1+\alpha_2} = kf_{22},$$
$$V_{-\alpha_1-\alpha_2} = kf_{11},\ V_{2\alpha_1+\alpha_2} = ke_{12},\ V_{-2\alpha_1-\alpha_2} = ke_{21}.$$

This way, T can be identified with the maximal torus in [**Hei**, page 13]: in Heinloth's description of the embedding $G \subset \mathrm{Sp}_6$ in characteristic 2, given by the action on

$$W = V/ke = W_{\alpha_1} \oplus W_{-\alpha_1-\alpha_2} \oplus W_{2\alpha_1+\alpha_2} \oplus W_{-2\alpha_1-\alpha_2} \oplus W_{\alpha_1+\alpha_2} \oplus W_{-\alpha_1},$$

the group G is generated by the two following copies of GL_2 :

$$\theta_1\colon A \longmapsto \begin{pmatrix} A & & \\ & A^{(1)} \det A^{-1} & \\ & & A \end{pmatrix} \quad \text{and} \quad \theta_2\colon B \longmapsto \begin{pmatrix} \det B^{-1} & & & \\ & B & & \\ & & B & \\ & & & \det B \end{pmatrix},$$

where $A^{(1)}$ denotes the Frobenius twist applied to A.

Lemma 3.2.12. *When considering the action of G on $\mathbf{P}(V) = \mathbf{P}^6$, we have*

$$\mathrm{Stab}_G([V_{2\alpha_1+\alpha_2}]) = P^{\alpha_1}.$$

Proof. First, let us prove that P^{α_1}, which is generated by T, $U_{\pm\alpha_2}$ and U_{α_1}, fixes $V_{2\alpha_1+\alpha_2} = ke_{12}$. Clearly the torus does; moreover, the computation of the respective actions of $u_{-\alpha_2}(\lambda)$, $u_{\alpha_2}(\lambda)$ and $u_{\alpha_1}(\lambda)$ on V, done in Chapter 6, Remark 6.2.1, shows that all three fix

$$[e_{12}] = [0\colon 0\colon 1\colon 0\colon 0\colon 0\colon 0].$$

This proves that $P^{\alpha_1} \subset S := \mathrm{Stab}_G([V_{2\alpha_1+\alpha_2}])$. To prove the reverse inclusion, let us remark that no nontrivial subgroup of $U_{-\alpha_1}$ and of $U_{-2\alpha_1-\alpha_2}$ fixes $[e_{12}]$: again by Chapter 6, Remark 6.2.1, we have

$$u_{-\alpha_1}(\lambda)\cdot e_{12} = e_{12} + \lambda f_{22} \quad \text{and} \quad u_{-2\alpha_1-\alpha_2}(\lambda)\cdot e_{12} = e_{12} + \lambda^2 e_{21},$$

thus $U_{-\alpha_1} \cap S = U_{-2\alpha_1-\alpha_2} \cap S = 1$.
At this point, we know that $\mathrm{Lie}\, P^{\alpha_1} \subset S$, hence by Corollary 3.2.8, $\mathrm{Lie}\, S$ is either equal to $\mathrm{Lie}\, P^{\alpha_1}$, to $\mathfrak{h}$, to $\mathfrak{l}$ or to $\mathrm{Lie}\, G$. However, $U_{\alpha_1} \cap S = 1$ means $\mathfrak{g}_{-\alpha_1}$ is not contained in $\mathrm{Lie}\, S$, hence the latter cannot be equal to $\mathfrak{l}$ nor to $\mathrm{Lie}\, G$. Analogously, $U_{-2\alpha_1-\alpha_2} \cap S = 1$ means $\mathfrak{g}_{2\alpha_1+\alpha_2}$ is not contained in $\mathrm{Lie}\, S$, hence $\mathrm{Lie}\, S$ cannot be equal to $\mathfrak{h}$. This means that $\mathrm{Lie}\, S = \mathrm{Lie}\, P^{\alpha_1}$ hence $S = P^{\alpha_1}$ as wanted. ☐

We can now conclude part of the proof of Proposition 3.2.11. First, let us recall that we are working with the basis $(f_{12}, f_{11}, e_{12}, e, e_{21}, f_{22}, f_{21})$ on V, giving homogeneous coordinates $[x_0\colon \cdots \colon x_6]$ on $\mathbf{P}(V)$: the norm q hence becomes

$$q(x) = x_3^2 + x_2x_4 + x_1x_5 + x_0x_6,$$

and its zero locus in $\mathbf{P}^6$ is the quadric Q of the Proposition. The point $[e_{12}]$ belongs to Q while $[e]$ does not, and the quotient $W = V/ke$ corresponds to the projection $\mathbf{P}^6\backslash\{[e]\} \longrightarrow \mathbf{P}^5$. Moreover, we have

$$G/P^{\alpha_1} = G/\mathrm{Stab}_G([V_{2\alpha_1+\alpha_2}]) = G\cdot[e_{12}] \hookrightarrow Q$$

Since both are smooth irreducible projective of dimension 5, they coincide. In particular,

$$\underline{\mathrm{Aut}}^0_{G/P^{\alpha_1}} = \underline{\mathrm{Aut}}^0_Q = \mathrm{SO}(V) = \mathrm{SO}_7$$

is of type B_3, as stated in Theorem 3.2.3.
What is left to prove is that $G/P_{\mathfrak{h}} \simeq \mathbf{P}^5$: to do this, we look at the action of G on W.

Lemma 3.2.13. *When considering the action of G on $\mathbf{P}(W) = \mathbf{P}^5$, we have*

$$\mathrm{Stab}_G([W_{2\alpha_1+\alpha_2}]) = P_{\mathfrak{h}}.$$

PROOF. Let S' be the stabilizer. From the above Lemma we know that P^{α_1} fixes $[V_{2\alpha_1+\alpha_2}]$, hence it also fixes $[W_{2\alpha_1+\alpha_2}]$. Moreover, by Chapter 6, Remark 6.2.1 we have

$$u_{-2\alpha_1-\alpha_2}(\lambda)\cdot[e_{12}] = [0\colon 0\colon 1\colon \lambda^2\colon 0\colon 0] \quad \text{and} \quad u_{-\alpha_1}(\lambda)\cdot[e_{12}] = [0\colon 0\colon 1\colon 0\colon \lambda\colon 0],$$

meaning that $U_{-\alpha_1}\cap S' = 1$, while

$$H = u_{-2\alpha_1-\alpha_2}(\alpha_p) = U_{-2\alpha_1-\alpha_2}\cap S'.$$

In particular, this yields that on one side, $P_{\mathfrak{h}} \subset S'$ hence $\mathfrak{h}\subset \operatorname{Lie} S$, and on the other side, $\mathfrak{g}_{-\alpha_1}$ is not contained in $\operatorname{Lie} S'$. In particular by Corollary 3.2.8 $\operatorname{Lie} S' = \mathfrak{h}$ and the only positive root γ satisfying $1 \subsetneq U_{-\gamma}\cap S' \subsetneq U_{-\gamma}$ is $-2\alpha_1-\alpha_2$, hence by [**Wen**, Proposition 8]

$$U^-_{S'} = \prod_{\gamma\in\Phi^+\colon U_{-\gamma}\not\subset S'} (U_{-\gamma}\cap S') = U_{-2\alpha_1-\alpha_2}\cap S' = H,$$

where U^-_P - following Wenzel's notation - denotes the infinitesimal unipotent subgroup given by the intersection of a parabolic subgroup P with the unipotent radical of the opposite of P_{red} with respect to the Borel B, as recalled in (2.3.1). Thus, we can conclude that $S' = U^-_{S'}\cdot S'_{\mathrm{red}} = H\cdot P^{\alpha_1}$, and the latter must coincide with $P_{\mathfrak{h}}$ by definition. □

Corollary 3.2.14. *We have $P_{\mathfrak{h}} = H\cdot P^{\alpha_1}$. More precisely,*

$$U^-_{P_{\mathfrak{h}}} = P_{\mathfrak{h}}\cap R_u((P^{\alpha_1})^-) = P_{\mathfrak{h}}\cap U_{-2\alpha_1-\alpha_2} = H.$$

Hence, $\operatorname{Lie} P_{\mathfrak{h}} = \mathfrak{h}$.

Now, let us consider the embedding

$$G/P_{\mathfrak{h}} = G/\operatorname{Stab}_G([W_{2\alpha_1+\alpha_2}]) = G\cdot[e_{12}] \hookrightarrow \mathbf{P}(W) = \mathbf{P}^5 \ .$$

As before, since both are smooth irreducible projective of dimension 5, they coincide. This gives as quotient map

$$G/P^{\alpha_1} = Q \longrightarrow G/(H\cdot P^{\alpha_1}) = \mathbf{P}^5 \tag{3.2.8}$$

the projection from $[e]$, which has degree 2 equal to the order of H.

3.2.3. Parabolic $P_{\mathfrak{l}}$. Let us consider the homogeneous space $G/P_{\mathfrak{l}}$ and show that one can realize it in a concrete way using octonions. More precisely, considering the action of $G_2\subset \mathrm{Sp}_6$ on $W = V/ke$, the parabolic subgroup $P_{\mathfrak{l}}$ is the stabilizer of a 3-dimensional isotropic vector subspace of W, spanned by the root spaces associated to the short positive roots (see Proposition 3.2.18). To do this, let us consider

$$\eta := f_{12}\wedge f_{22}\wedge e_{12}$$

as element of $\mathbf{P}(\Lambda^3 V)$ and $\overline{\eta}$ the element of $\mathbf{P}(\Lambda^3 W)$ given by the images in W of the three vectors.

Lemma 3.2.15. *Let G be simple of type G_2 in characteristic $p=2$ and $P_{\mathfrak{l}}$ the parabolic subgroup of Definition 3.2.9. When considering the action of G on $\mathbf{P}(\Lambda^3 V)$ and $\mathbf{P}(\Lambda^3 W)$ respectively, we have*

$$\operatorname{Stab}_G(\eta) = P^{\alpha_1} \quad \textit{and} \quad \operatorname{Stab}_G(\overline{\eta}) = P_{\mathfrak{l}}.$$

Proof. Let us denote as S and S'' the above stabilizers.
First, let us prove that P^{α_1}, which is generated by T, $U_{\pm\alpha_2}$ and U_{α_1}, fixes the subspace $kf_{12} \oplus kf_{22} \oplus ke_{12} \subset V$, whose elements are of the form $(w_0, 0, w_2, 0, 0, w_5, 0)$. The computations of Chapter 6, Remark 6.2.1 give us the following :

$$\begin{aligned} u_{\alpha_2}(\lambda) \cdot (w_0, 0, w_2, 0, 0, w_5, 0) &= (w_0, 0, w_2, 0, 0, \lambda w_0 + w_5, 0), \\ u_{-\alpha_2}(\lambda) \cdot (w_0, 0, w_2, 0, 0, w_5, 0) &= (w_0 + \lambda w_5, 0, w_2, 0, 0, w_5, 0) \\ u_{\alpha_1}(\lambda) \cdot (w_0, 0, w_2, 0, 0, w_5, 0) &= (w_0, 0, w_2 + \lambda w_5, 0, 0, w_5, 0), \end{aligned}$$

meaning that $P^{\alpha_1} \subset S$. Moreover, considering the action of the root subgroups associated to $-\alpha_1$, $-2\alpha_1 - \alpha_2$ and $-\alpha_1 - \alpha_2$, we have the following :

$$\begin{aligned} u_{-\alpha_1}(\lambda) \cdot (w_0, 0, w_2, 0, 0, w_5, 0) &= (w_0, 0, w_2, \lambda w_0, 0, \lambda w_2 + w_5, \lambda^2 w_0), \\ u_{-2\alpha_1-\alpha_2}(\lambda) \cdot (w_0, 0, w_2, 0, 0, w_5, 0) &= (w_0, \lambda w_0, w_2, \lambda w_2, \lambda^2 w_2, w_5, \lambda w_5), \\ u_{-\alpha_1-\alpha_2}(\lambda) \cdot (w_0, 0, w_2, 0, 0, w_5, 0) &= (w_0 + \lambda w_2, \lambda^2 w_5, w_2, \lambda w_5, 0, w_5, 0). \end{aligned}$$

These computations imply that $\operatorname{Lie} S$ has trivial intersection with the root subspaces associated to short negative roots. Thus by Corollary 3.2.8 $\operatorname{Lie} S = \operatorname{Lie} P^{\alpha_1}$, which allows us to conclude that $S = P^{\alpha_1}$.
Next, let us consider the action of G on the quotient $W = V/ke$. The second computation just above yields that the intersection $U_{-2\alpha_2-\alpha_1} \cap S''$ is trivial, hence $\mathfrak{g}_{-2\alpha_1-\alpha_2}$ is not contained in $\operatorname{Lie} S''$ and the latter cannot be equal to $\operatorname{Lie} G$ nor to $\mathfrak{h}$. The other two equalities imply that $U_{-\alpha_1} \cap S'' = u_{-\alpha_1}(\alpha_p)$ and $U_{-\alpha_1-\alpha_2} \cap S'' = u_{-\alpha_1-\alpha_2}(\alpha_p)$, meaning that $\operatorname{Lie} S'' = \mathfrak{l}$. In particular, the positive roots γ satisfying $1 \subsetneq U_{-\gamma} \cap S' \subsetneq U_{-\gamma}$ are α_1 and $\alpha_1 + \alpha_2$: by [Wen], we have

$$U^-_{S''} = \prod_{\gamma \in \Phi^+ :\, U_{-\gamma} \not\subset S''} (U_{-\gamma} \cap S'') = (U_{-\alpha_1-\alpha_2} \cap S'') \cdot (U_{-\alpha_1} \cap S'') = L.$$

Thus, we can conclude that $S'' = U^-_{S''} \cdot S''_{\mathrm{red}} = L \cdot P^{\alpha_1}$, and the latter must coincide with $P_{\mathfrak{l}}$ by definition. □

Corollary 3.2.16. *We have $P_{\mathfrak{l}} = L \cdot P^{\alpha_1}$. More precisely,*

$$U^-_{P_{\mathfrak{l}}} = P_{\mathfrak{l}} \cap R_u((P^{\alpha_1})^-) = (P_{\mathfrak{l}} \cap U_{-\alpha_1-\alpha_2}) \cdot (P_{\mathfrak{l}} \cap U_{-\alpha_1}) = L.$$

Hence, $\operatorname{Lie} P_{\mathfrak{l}} = \mathfrak{l}$.

Next, let us realise the variety Q as a hyperplane section of the SO_7-homogeneous variety of isotropic 3-dimensional subspaces of V: this will help us describe $X := G/P_{\mathfrak{l}}$ geometrically. Recall that - keeping the notation from Proposition 3.1.12 - the reduced parabolic subgroup associated to the short root α_3 in type B_3, which is denoted $P_3 = P^{\alpha_3} \subset \mathrm{SO}_7$, is the stabilizer of an isotropic subspace of dimension 3, hence

$$P^{\alpha_1} = \operatorname{Stab}_G(\eta) = G \cap P_3 = G \cap \operatorname{Stab}_{\mathrm{SO}_7}(\eta).$$

This gives the following embedding, where we denote as $\mathcal{L}$ the unique (very) ample generator of the Picard group of Y.

$$Q = G/P^{\alpha_1} \hookrightarrow Y := \mathrm{SO}_7/P_3 \hookrightarrow \mathbf{P}(H^0(Y, \mathcal{L})^\vee) \tag{3.2.9}$$

Lemma 3.2.17. *The variety Q is a hyperplane section of Y relative to the ample line bundle $\mathcal{L}$.*

Proof. Let us express Y as a quotient of Spin_7 by the maximal reduced parabolic Q_3 associated to the short simple root. Since Spin_7 is simply connected, the Picard group of Y identifies with the group of characters of Q_3. Under this identification, the embedding (3.2.9) is given by the representation of Spin_7 acting on $U := H^0(Y, \mathcal{L})$, whose associated weight ϖ is the third fundamental weight in type B_3. This weight is minuscule, so the set of weights of U is acted on transitively by the Weyl group. This gives that the weights of the diagonal maximal torus (3.1.4) of SO_7 in U are

$$\frac{1}{2}\left(\pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3\right). \tag{3.2.10}$$

In particular, U has dimension 8, so that (3.2.9) is a codimension one embedding of Y into $\mathbf{P}(U^\vee)$. Moreover, by [**Ram**, Theorem 3.11], the homogeneous ideal of Y is generated by degree 2 elements, hence there is some non-degenerate quadratic form q on U of which Y is the zero locus.
Next, let us restrict the representation to G: the maximal torus T we consider is the one given in (6.1.4), hence (3.2.10) gives as T-weights of U the six short roots

$$\pm\alpha_1, \quad \pm(\alpha_1+\alpha_2), \quad \pm(2\alpha_1+\alpha_2),$$

together with twice the zero weight. In particular, U admits, as a G-module, only two irreducible quotients, the trivial representation and the simple module W which has as weights the six short roots. Moreover, the quadratic form q provides an isomorphism between U and its dual as G modules: in particular, there exists some linear form h on U invariant by G. Since the base point of Q corresponds to a B-stable line in U with weight ϖ, h must vanish on it and therefore there is an inclusion of Q into the hyperplane $H = (h = 0)$. Finally, the intersection $H \cap Y$ has dimension 5, contains Q and is a hypersurface because h is linear and q non-degenerate, hence it must coincide with Q and we are done. ☐

The above description of the variety Q holds in any characteristic. The case of characteristic two is peculiar because there exists an embedding of G_2 into Sp_6, together with the very special isogeny described in Example 2.5.16. We will now use these two ingredients to get a geometric description of X, starting from the above realisation of the variety Q and the natural quotient morphism $Q \to X$, induced by the inclusion of $P^{\alpha_1} = (P_{\mathfrak{l}})_{\mathrm{red}}$ into $P_{\mathfrak{l}}$.

Let us consider the following commutative diagram, which is induced by the quotient $W = V/ke$ and the associated purely inseparable isogeny

$$\varphi\colon\ \mathrm{SO}(V) = \mathrm{SO}_7 \longrightarrow \mathrm{Sp}_6 = \mathrm{Sp}(W),$$

with kernel $N := N_{\mathrm{SO}_7}$. Let us recall that, by Lemma 3.2.15, Q is the G_2-orbit of the 3-dimensional subspace defined by the short positive root vectors in $\Lambda^3 V$, while X is the G_2-orbit of the 3-dimensional subspace defined by the short positive root vectors in $\Lambda^3 W$.

$$\begin{array}{ccc} Q = G_2/P^{\alpha_1} & \xrightarrow{\quad g \quad} & X = G_2/P_{\mathfrak{l}} \\ \big\downarrow & & \big\downarrow \\ Y := \mathrm{SO}_7/P_3 & \xrightarrow{\quad f \quad} & Z := \mathrm{Sp}_6/P'_3 = \mathrm{SO}_7/(NP_3) \\ \big\downarrow & & \big\downarrow \\ \mathbf{P}(\Lambda^3 V) & \dashrightarrow & \mathbf{P}(\Lambda^3 W) \end{array}$$

Proposition 3.2.18. *The line bundle* $\mathcal{O}_Z(X)$ *satisfies the equality* $\operatorname{Pic} Z = \mathbf{Z}\,\mathcal{O}_Z(X)$*. In particular,* X *is a hyperplane section of* Z *with respect to the unique (very) ample generator of* $\operatorname{Pic} Z$*.*

Proof. By Lemma 3.2.17, the Picard group of Y is generated by $\mathcal{O}_Y(Q)$, hence Q satisfies

$$Q \cdot \widetilde{C} = 1,$$

where we denote respectively as $\widetilde{C}$ and C the Schubert curves (associated to the short simple root α_3 in type B_3 and the long simple root α'_3 in type C_3; see Section 5.1 for a detailed definition) in Y and in Z.
The morphism f is finite locally free of degree 8, which corresponds to the order of

$$NP_3/P_3 = N/(N \cap P_3).$$

Indeed, as seen in Example 2.5.16, the subgroup $N \subset \mathrm{SO}_7$ has height one and Lie algebra $\mathfrak{n} = \mathfrak{g}_<$ of dimension 6, hence the order of N is 2^6. On the other hand, the order of $N \cap P_3$ is 2^3 because

$$\operatorname{Lie}(N \cap P_3) = \mathfrak{n} \cap \operatorname{Lie} P_3 = \mathfrak{g}_{-\varepsilon_1-\varepsilon_2} \oplus \mathfrak{g}_{-\varepsilon_1-\varepsilon_3} \oplus \mathfrak{g}_{-\varepsilon_2-\varepsilon_3}$$

has dimension 3. In particular, this means that $f_*f^*X = 8X$ seen as elements of $\operatorname{Pic} Z$.
On the other hand, g is finite locally free of degree 4: the latter is the order of L, the unipotent infinitesimal part of $P_{\mathfrak{l}}$. Thus we also have $f_*Q = 4X$: putting the two equalities together implies $f^*X = 2Q$ in the Picard group of Y.
Next we notice that α_3 is a short root in type B_3, hence the very special isogeny acts as a Frobenius morphism on the corresponding copy of the additive group in SO_7. In other words, the set theoretic equality $f(\widetilde{C}) = C$ becomes $f_*\widetilde{C} = 2C$ on 1-cycles. In particular,

$$2 = 2Q \cdot \widetilde{C} = f^*X \cdot \widetilde{C} = X \cdot f_*\widetilde{C} = 2X \cdot C.$$

This last computation together with the fact that $\operatorname{Pic} Z \simeq \mathbf{Z}$ allows us to conclude that the line bundle associated to X generates the Picard group of Z. □

Up to this point we have realized the variety $X = G/P_{\mathfrak{l}}$ using octonions. In particular, this construction provides a new example (besides projective spaces and quadrics) of a hyperplane section X of a homogeneous variety $(Z, \mathcal{M})$, such that X is also homogeneous and $\mathcal{M}$ generates the Picard group of Z. One might ask whether Theorem 3.1.1 still holds for the variety X. Actually this is not the case, as illustrated in the following result.

Proposition 3.2.19. *Let* G *be simple of type* G_2 *in characteristic* $p = 2$ *and* $P_{\mathfrak{l}}$ *the parabolic subgroup of Definition 3.2.9. Then* $G/P_{\mathfrak{l}}$ *is not isomorphic, as a variety, to a quotient of the form* G'/P^{α} *for any* G' *simple and* $\alpha \in \Delta(G')$*.*

In particular, this means that Theorem 3.1.1 does not hold in this case. The first step in the proof of Proposition 3.2.19 is the following.

Lemma 3.2.20. *Let G' be simple and let α be a simple root of G'. If $\dim(G'/P^\alpha) = 5$, then such a variety is either isomorphic to $Q \subset \mathbf{P}^6$, to $\mathbf{P}^5$ or to G/P^{α_2} where G is of type G_2 and α_2 is the long root.*

Proof. Let us recall that

$$dim(G'/P^\alpha) = |\Phi^+(G)| - |\Phi^+(L^\alpha)|,$$

where $L^\alpha = P^\alpha \cap (P^\alpha)^-$ is a Levi subgroup, hence so we can compute this quantity explicitly in each case.

Type A_{n-1}: for $1 \leq m \leq n-1$,

$$\dim(G'/P^{\alpha_m}) = m(n-m) = 5$$

when $(n, m) = (6, 5)$ or $(6, 1)$. In that case, $G'/P^{\alpha_1} = G'/P^{\alpha_5} \simeq \mathbf{P}^5$.

Type B_n: the number of positive roots is n^2.

• For $1 \leq m \leq n-1$, the Levi subgroup $P^{\alpha_m} \cap (P^{\alpha_m})^-$ is of type $A_{m-1} \times B_{n-m}$, so

$$\dim(G'/P^{\alpha_m}) = n^2 - \frac{m(m-1)}{2} - (n-m)^2 = m\left(\frac{1-m}{2} + 2n - m\right) = 5$$

which only has as positive integer solutions the pairs $(n, m) = (4, 5)$, which is absurd, and $(n, m) = (3, 1)$. In that case, $G' = \mathrm{SO}_7$ and by Theorem 3.2.3 and Proposition 3.2.11 we have $\mathrm{SO}_7 / P^{\alpha_1} \simeq G/P^{\alpha_1} \simeq Q \subset \mathbf{P}^6$.

• Considering the last simple root, $P^{\alpha_n} \cap (P^{\alpha_n})^-$ is of type A_{n-1} and

$$\dim(G'/P^{\alpha_n}) = n^2 - \frac{n(n-1)}{2} = \frac{n(n+1)}{2}$$

is never equal to 5.

Type C_n: the same computations as in type B_n give $(n, m) = (3, 1)$, meaning $G' = \mathrm{PSp}_6$ and - again by Theorem 3.2.3 - we have $\mathrm{PSp}_6 / P^{\alpha_1} = \mathrm{PSL}_6 / P^{\alpha_1} \simeq \mathbf{P}^5$.

Type D_n: the number of positive roots is $n(n-1)$.

• For $1 \leq m \leq n-4$, the Levi subgroup is of type $A_{m-1} \times D_{n-m}$, so

$$\dim(G'/P^{\alpha_m}) = n(n-1) - \frac{m(m-1)}{2} - (n-m)(n-m-1) = m\left(\frac{1-m}{2} + 2n - m - 1\right) = 5$$

which has no positive integer solutions (n, m).

• For $m = n-3$, the Levi subgroup is of type $A_{n-4} \times A_3$, so

$$\dim(G'/P^{\alpha_m}) = n(n-1) - \frac{(n-3)(n-4)}{2} - 6 = 5,$$

which gives $n^2 + 5n = 34$ hence no integer solutions.

• For $m = n-2$, the Levi subgroup is of type $A_{n-3} \times A_1 \times A_1$, so

$$\dim(G'/P^{\alpha_m}) = n(n-1) - \frac{(n-2)(n-3)}{2} - 1 - 1 = 5,$$

which gives $n^2+3n=20$ hence no integer solutions.

- For $m=n-1$ or $m=n$, the Levi subgroup is of type A_{n-1}, so

$$\dim(G'/P^{\alpha_m}) = n(n-1) - \frac{n(n-1)}{2} = \frac{n(n-1)}{2},$$

which is never equal to 5.

Type E_6: the number of positive roots is 36, and the following table

E_6	α_1	α_2	α_3	α_4	α_5	α_6
L^α	D_5	$A_4\times A_1\times A_1$	$A_2\times A_2\times A_1$	$A_4\times A_1$	D_5	A_5
$\lvert\Phi^+(L^\alpha)\rvert$	20	11	7	11	20	15
$\dim(G/P^\alpha)$	16	25	29	25	16	21

shows that the desired quantity is never equal to 5.

Type E_7: the number of positive roots is 63 and the following table

E_7	α_1	α_2	α_3	α_4	α_5	α_6	α_7
L^α	D_6	$A_5\times A_1$	$A_1\times A_2\times A_3$	$A_4\times A_2$	$D_5\times A_1$	E_6	A_6
$\lvert\Phi^+(L^\alpha)\rvert$	30	16	10	13	21	36	21
$\dim(G/P^\alpha)$	33	47	53	50	42	27	42

shows that the desired quantity is never equal to 5.

Type E_8: the number of positive roots is 120 and the following table

E_8	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8
L^α	D_7	$A_6\times A_1$	$A_1\times A_2\times A_4$	$A_4\times A_3$	$D_5\times A_2$	$E_6\times A_1$	E_7	A_7
$\lvert\Phi^+(L^\alpha)\rvert$	42	22	14	16	23	37	63	28
$\dim(G/P^\alpha)$	78	98	106	104	97	83	57	92

shows that the desired quantity is never equal to 5.

Type F_4: a direct computation - see Section 3.1.5 - gives

$$\dim(G'/P^{\alpha_1}) = \dim(G'/P^{\alpha_4}) = 15 \quad \text{and} \quad \dim(G'/P^{\alpha_2}) = \dim(G'/P^{\alpha_3}) = 20.$$

Type G_2: as we already know, both $G/P^{\alpha_1}=Q$ and G/P^{α_2} have dimension 5. □

Lemma 3.2.21. *The variety $X=G/P_\mathfrak{l}$ is not isomorphic to $\mathbf{P}^5$ nor to Q.*

PROOF. Let us consider the quotient map $f\colon G/P^{\alpha_1}\longrightarrow G/P_\mathfrak{l}$. By Corollary 3.2.16 we have $P_\mathfrak{l}=L\cdot P^{\alpha_1}$, hence the morphism f is finite, purely inseparable and of degree 4. Assume $X\simeq \mathbf{P}^5$, then we get $f\colon Q\longrightarrow \mathbf{P}^5$. Considering the line bundle $\mathcal{O}_Q(1)=\mathcal{O}_{\mathbf{P}^6}(1)_{|Q}$, we have that $\operatorname{Pic} Q=\mathbf{Z}\cdot\mathcal{O}_Q(1)$ and $f^*\mathcal{O}_{\mathbf{P}^5}(1)=\mathcal{O}_Q(m)$ for some $m>0$, since it has sections. Taking degrees, this gives on the left hand side

$$\begin{aligned}&f^*\mathcal{O}_{\mathbf{P}^5}(1)\cdot f^*\mathcal{O}_{\mathbf{P}^5}(1)\cdot f^*\mathcal{O}_{\mathbf{P}^5}(1)\cdot f^*\mathcal{O}_{\mathbf{P}^5}(1)\cdot f^*\mathcal{O}_{\mathbf{P}^5}(1)\\ =&(\deg f)\,(\mathcal{O}_{\mathbf{P}^5}(1)\cdot\mathcal{O}_{\mathbf{P}^5}(1)\cdot\mathcal{O}_{\mathbf{P}^5}(1)\cdot\mathcal{O}_{\mathbf{P}^5}(1)\cdot\mathcal{O}_{\mathbf{P}^5}(1))=\deg f,\end{aligned}$$

so we get $\deg f=4$. On the right hand side, this equals

$$\begin{aligned}&\mathcal{O}_Q(m)\cdot\mathcal{O}_Q(m)\cdot\mathcal{O}_Q(m)\cdot\mathcal{O}_Q(m)\cdot\mathcal{O}_Q(m)\\ &=\rho^*\mathcal{O}_{\mathbf{P}^5}(m)\cdot\rho^*\mathcal{O}_{\mathbf{P}^5}(m)\cdot\rho^*\mathcal{O}_{\mathbf{P}^5}(m)\cdot\rho^*\mathcal{O}_{\mathbf{P}^5}(m)\cdot\rho^*\mathcal{O}_{\mathbf{P}^5}(m)\\ &=(\deg\rho)(\mathcal{O}_{\mathbf{P}^5}(m)\cdot\mathcal{O}_{\mathbf{P}^5}(m)\cdot\mathcal{O}_{\mathbf{P}^5}(m)\cdot\mathcal{O}_{\mathbf{P}^5}(m)\cdot\mathcal{O}_{\mathbf{P}^5}(m))=(\deg\rho\cdot m^5),\end{aligned}$$

which has degree $2m^5$, where ρ is the projection of Proposition 3.2.11. Comparing degrees one gets $4 = 2m^5$, which is absurd.

Now, let us assume instead that $X \simeq Q$, then $f\colon Q \longrightarrow Q$ is of degree 4 and again $f^*\mathcal{O}_Q(1) = \mathcal{O}_Q(r)$ for some $r > 0$: the analogous computation of degrees yields $8 = 2r^5$, which is again absurd. □

Lemma 3.2.22. *The variety $X = G/P_{\mathfrak{l}}$ is not isomorphic to G/P^{α_2}. Thus, Proposition 3.2.19 holds.*

Proof. Assume $X \simeq G/P^{\alpha_2}$, then the G-action on X is given by a morphism $\theta\colon G \to \underline{\mathrm{Aut}}^0_{G/P^{\alpha_2}}$, the latter being equal to G by Theorem 3.2.3. In particular, θ is an isogeny which satisfies $\theta^{-1}(P^{\alpha_2}) = P_{\mathfrak{l}}$. This means that there is some $g \in G(k)$ such that

$$(\ker\theta)\cdot gP^{\alpha_2}g^{-1} = P_{\mathfrak{l}}.$$

Since $\ker\theta$ is finite, taking the connected component of the identity and the reduced subscheme on both sides implies that P^{α_2} and P^{α_1} are conjugate in G, which is a contradiction. □

The above study of $P_{\mathfrak{h}}$ and $P_{\mathfrak{l}}$ does not complete the classification (in characteristic 2) of homogeneous spaces having as stabilizer a parabolic subgroup whose reduced part is equal to P^{α_1}. Let us consider a simple group G of type G_2 and a nonreduced parabolic subgroup $P \subset G$ satisfying $P_{\mathrm{red}} = P^{\alpha_1}$, in characteristic $p = 2$. Moreover, let us assume that $\mathrm{Lie}\,P \neq \mathrm{Lie}\,G$, i.e. that $\mathrm{Lie}\,P$ is equal to $\mathfrak{h}$ (resp. $\mathfrak{l}$) and let us write $P = U_P^- \cdot P_{\mathrm{red}}$, where $U_P^- = P \cap R_u(P^-_{\mathrm{red}})$: in particular, the unipotent infinitesimal subgroup U_P^- is contained in $U_{-2\alpha_1-\alpha_2}$ (resp. in $U_{-\alpha_1}\cdot U_{-\alpha_1-\alpha_2}$) and its order is $|U_P^-| = 2^n$ for some $n \geq 2$, the case $n = 1$ being $P_{\mathfrak{h}}$ treated above.

3.2.4. End of classification in type G_2. Recall that we follow here the notation from [Wen] : for a parabolic subgroup P, we denote as U_P^- the intersection of P with the unipotent radical of the opposite of P_{red}.

Lemma 3.2.23. *Let P be a parabolic subgroup such that* $\mathrm{Lie}\,P = \mathfrak{h}$. *Then its unipotent infinitesimal part U_P^- has height one.*

Proof. The reduced part of P is P^{α_1}, hence U_P^- must be of the form $u_{-2\alpha_1-\alpha_2}(\alpha_{p^n})$ for some n. Let us assume that n is at least equal to 2. This means that there is some $\lambda \in \mathbf{G}_a$ such that $\lambda^2 \neq 0$ and $u_{-2\alpha_1-\alpha_2}(\lambda) \in P$. Let us consider $\mu \in \mathbf{G}_a$ and compute the following commutator, which gives an element of P:

$$(u_{-2\alpha_1-\alpha_2}(\lambda), u_{\alpha_1}(\mu)) = u_{-2\alpha_1-\alpha_2}(\lambda)u_{\alpha_1}(\mu)u_{-2\alpha_1-\alpha_2}(-\lambda)u_{\alpha_1}(-\mu) = (u_{-2\alpha_1-\alpha_2}(\lambda)u_{\alpha_1}(\mu))^2$$

$$= \left(\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \lambda & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \lambda^2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \lambda & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \mu^2 \\ 0 & 1 & 0 & \mu & 0 & 0 \\ 0 & 0 & 1 & 0 & \mu & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}\right)^2 = \begin{pmatrix} 1 & 0 & 0 & 0 & \lambda\mu^2 & 0 \\ 0 & 1 & \mu\lambda^2 & 0 & 0 & \lambda\mu^2 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \mu\lambda^2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The last quantity, when assuming $\mu^2 = 0$, coincides with $u_{-3\alpha_1-2\alpha_2}(\mu\lambda^2)$, which is a contradiction with the fact that $\operatorname{Lie} P = \mathfrak{h}$ does not intersect the root subspace associated to the root $-3\alpha_1 - 2\alpha_2$. □

Lemma 3.2.24. *Let P be a parabolic subgroup such that $\operatorname{Lie} P = \mathfrak{l}$. Then its unipotent infinitesimal part U_P^- has height one.*

Proof. As before, the reduced part of P is P^{α_1}. Moreover, the unipotent part U_P^- has nontrivial and finite intersection with $U_{-\alpha_1}$ and $U_{-\alpha_1-\alpha_2}$, of height m_1 and m_2 respectively. Assuming the height of U_P^- to be at least equal to 2 means we have (up to a reflection by s_{α_2}) that $m_2 \geq 2$. Thus, let $\lambda \in \mathbf{G}_a$ such that $\lambda^2 \neq 0$ and $\mu \in \alpha_p$, so that $u_{-\alpha_1}(\mu) \in P$. Then the following commutator also belongs to P :

$$(u_{-\alpha_1-\alpha_2}(\lambda), u_{-\alpha_1}(\mu)) = u_{-\alpha_1-\alpha_2}(\lambda)u_{-\alpha_1}(\mu)u_{-\alpha_1-\alpha_2}(-\lambda)u_{-\alpha_1}(-\mu) = (u_{-\alpha_1-\alpha_2}(\lambda)u_{-\alpha_1}(\mu))^2$$

$$= \left(\begin{pmatrix} 1 & 0 & \lambda & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \lambda^2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \lambda \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \mu & 0 & 1 & 0 & 0 \\ 0 & 0 & \mu & 0 & 1 & 0 \\ \mu^2 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right)^2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & \mu\lambda^2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \lambda\mu^2 & 0 & 0 & 1 & \mu\lambda^2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \lambda\mu^2 & 0 & 0 & 1 \end{pmatrix}.$$

The last quantity coincides again with $u_{-3\alpha_1-2\alpha_2}(\mu\lambda^2)$, so we conclude as before. □

Definition 3.2.25. For an integer $m \geq 0$, we denote as ${}_mH$ and ${}_mL$ the pull-back respectively of the subgroups H and L under an m-th iterated Frobenius morphism.

Proposition 3.2.26. *Let G be of type G_2 in characteristic two.*
Then the nonreduced parabolic subgroups of G having P^{α_1} as reduced part are all of the form ${}_mGP^{\alpha_1}$, ${}_mHP^{\alpha_1}$ or ${}_mLP^{\alpha_1}$ for some $m \geq 0$.

Proof. Let us consider such a subgroup P: its Lie algebra contains strictly $\operatorname{Lie} P^{\alpha_1}$, hence by Corollary 3.2.8 it is either equal to $\operatorname{Lie} G$, to $\mathfrak{h}$ or to $\mathfrak{l}$. If $\operatorname{Lie} P = \operatorname{Lie} G$, then there is a unique integer $m \geq 1$ such that the Frobenius kernel ${}_mG$ is contained in P while ${}_{m+1}G$ is not. Considering the quotient $P' := P/{}_mG$ allows us to assume that the Lie algebra of P' is strictly contained in the one of G. Next, if $\operatorname{Lie} P' = \mathfrak{h}$ (resp. $\mathfrak{l}$), by Lemma 3.2.23 and Lemma 3.2.24, we have that $P' = P_{\mathfrak{h}}$ (resp. $P_{\mathfrak{l}}$). Thus, the parabolic P is obtained from P^{α_1}, $P_{\mathfrak{h}}$ or $P_{\mathfrak{l}}$ by pulling back with an iterated Frobenius morphism, and we are done. □

This completes the proof of Theorem 3.2.1 and thus gives a complete classification of homogeneous varieties with Picard group $\mathbf{Z}$, which ends the proof of Theorem 3.1.1.

Remark 3.2.27. The last result, together with Proposition 3.2.18, has as consequence the fact that any ample line bundle on an homogeneous variety of Picard rank one is very ample, without any assumption of type nor characteristic. This will be generalized in Corollary 5.4.3 to arbitrary Picard ranks.

Remark 3.2.28. Let us cite a reason why the geometry of a general projective homogeneous variety of Picard rank one may differ from the one of a generalized flag variety. This comes from the following generalization of a question of Lazarsfeld (see the end of [Laz]): if $X = G/P$ has Picard group isomorphic to $\mathbf{Z}$ and there is some surjective morphism $f\colon X \to Y$, then is Y isomorphic to X? First, the iterated Frobenius morphisms $G/P \to G/{}_mGP$ do not give a counterexample. However, the maps

$$G/P^\alpha \longrightarrow G/N_G P^\alpha \quad \text{and} \quad G_2/P^{\alpha_1} \longrightarrow G_2/P_{\mathfrak{l}},$$

defined respectively under the edge hypothesis and in characteristic 2, are counterexamples. Both these examples are purely inseparable surjective morphisms: the next natural step would be adding the hypothesis for the morphism f to be generically étale.

3.3. End of classification

We state here - in all types but G_2 - the desired modification of Wenzel's description of parabolic subgroups having as reduced subgroup a maximal one: they are all obtained by fattening the reduced part with the kernel of a noncentral isogeny, which generalizes to this setting the role of the Frobenius in characteristic $p \geq 5$. We then give a criterion to determine when two homogeneous spaces with Picard rank one have the same underlying variety.

3.3.1. Consequences in rank one. We complete here the study in the case of Picard rank one. Due to Proposition 3.2.19, let us make the assumption that the group G is not of type G_2 in characteristic two.

3.3.1.1. *Classification, quasi-standard type.* The results in the preceding Section allow us to complete the classification of parabolic subgroups having as reduced subgroup a maximal one. Let us recall that, by [Wen], if the Dynkin diagram of G is simply laced or if $p \geq 5$, then such subgroups are of the form

$${}_mGP^\alpha = (\ker F_G^m)P^\alpha.$$

Example 3.3.1. Before moving on to state the classification, let us mention the very first examples that were constructed of two parabolic subgroups which have maximal reduced part and are not of standard type (meaning, of the form just above). These are called *exceptional* parabolic subgroups in [Lau2, Section 3.3]. In this article, he constructs these examples by means of representation theory methods; we are able here to give a more concise description. Let us place ourselves over a field of characteristic $p = 2$. The first example is the following: in a group of type B_2, with short simple root α, the parabolic

$$NP^\alpha$$

is described, where N denotes the kernel of the very special isogeny. The second example, on the other hand, is the parabolic subgroup

$$NP^\beta,$$

where β is the long simple root, in a group of type C_4.

THEOREM 3.3.2. *Let G be simple and P be a non-reduced parabolic subgroup of G such that its reduced subgroup is maximal i.e. of the form $P_{red} = P^\alpha$ for some simple root α. Then there exists an isogeny φ with source G such that*

$$P = (\ker \varphi)P^\alpha,$$

unless G is of type G_2 over a field of characteristic $p = 2$ and α is the simple short root.

PROOF. First, Proposition 3.1.9, Proposition 3.1.10, Proposition 3.2.4, Proposition 3.1.16, Remark 3.1.13 and Remark 3.1.15 imply that if G is simple and P_{red} is a maximal reduced parabolic subgroup, then either P is reduced, or there exists a nontrivial noncentral normal subgroup of height one contained in P. This subgroup is either $H = N_G$ - when it is defined - or the image of the Frobenius kernel of the simply connected cover of G.
Now, let us consider the given parabolic P. If it is reduced, then there is nothing to prove. If it is nonreduced, then there is a noncentral subgroup $H_{(1)} \subset P$ normalized by G and of height one. Let us denote as

$$\varphi_1 \colon G \longrightarrow G/H_{(1)} =: G_{(1)}$$

the quotient morphism and replace the pair (G, P) with $(G_{(1)}, P_{(1)})$, where $P_{(1)} := P/H_{(1)}$. This gives again a parabolic subgroup whose reduced subgroup is maximal, hence either $P_{(1)}$ is reduced or we can repeat the same reasoning to get an isogeny

$$\varphi_2 \colon G \longrightarrow G/H_{(1)} \longrightarrow G/H_{(2)} =: G_{(2)}.$$

Setting $P_{(2)} := G/H_{(2)}$ we repeat the same reasoning again. This gives a sequence $(G_{(m)}, P_{(m)})$ which ends with a reduced parabolic subgroup in a finite number of steps : indeed, P/P_{red} is finite so it is not possible to have an infinite sequence

$$P_{\text{red}} \subsetneq H_{(1)}P_{\text{red}} \subsetneq \cdots \subsetneq H_{(m)}P_{\text{red}} \subsetneq \cdots \subsetneq P.$$

Thus, let us set $H := H_{(m)}$ for m big enough and $\varphi := \varphi_m$. Then we claim that $P = HP^\alpha = (\ker \varphi)P^\alpha$.
Both H and P^α are subgroups of P by construction, hence $HP^\alpha \subset P$. Quotienting by H then gives

$$HP^\alpha/H = P^\alpha/(H \cap P^\alpha) \subset P/H = P_{(m)}.$$

Since both are reduced and have the same underlying topological space, they must coincide hence $HP^\alpha = P$. □

In particular, using our previous results on factorisation of isogenies, we can give a very explicit description of the kernels involved in the classification.

Corollary 3.3.3. *Keeping the above notation and the ones given in Definition 2.5.8, in the equality $P = (\ker \varphi)P^\alpha$, there are only the two following options:*

(a) *either* $\ker \varphi = \ker F^m_G = {}_mG$ *is the Frobenius kernel,*
(b) *or, when such a subgroup is defined,* $\ker \varphi = \ker(\pi_{G^{(m)}} \circ F^m_G) = {}_mN_G$.

PROOF. Let us first assume G to be simply connected and consider the factorisation of the isogeny φ given by Proposition 2.5.12

$$\varphi \colon G \xrightarrow{\ \sigma\ } G'' \xrightarrow{\ \rho\ } G',$$

where $\sigma = \pi \circ F^m$ and ρ is central. Let α, α'' and α' be simple roots of G, G'' and G' respectively, defined by the equalities

$$P_{\text{red}} = P^\alpha, \quad \sigma(P^\alpha) = P^{\alpha''}, \quad \rho(P^{\alpha''}) = P^{\alpha'}.$$

Then

$$P = (\ker \rho\sigma) P^\alpha = (\rho\sigma)^{-1}(P^{\alpha'}) = \sigma^{-1}(P^{\alpha''}) = (\ker \sigma) P^\alpha,$$

hence replacing φ by σ and G' by G'' gives one of the cases (a) and (b).
If G is not simply connected, then we can consider the pull-back $\widetilde{P} := \psi^{-1}(P) \subset \widetilde{G}$ in the simply connected cover. Applying the above reasoning to $\widetilde{P}$ yields

$$\text{either } P = \psi(\widetilde{P}) = \psi({}_m\widetilde{G}P^\alpha) = {}_mGP^\alpha, \quad \text{or } P = \psi(\widetilde{P}) = \psi({}_mN_{\widetilde{G}}P^\alpha) = {}_mN_GP^\alpha$$

and we are done. □

3.3.1.2. *Comparing varieties of Picard rank one.* Let us start by considering a homogeneous variety $X = G/P$ under the action of a simple adjoint group G, having Picard group of rank one. Then set

$$G_0 := \underline{\mathrm{Aut}}^0_X \quad \text{and} \quad P_0 := \mathrm{Stab}(x) \subset G_0,$$

where $x \in X$ is a closed point and where we keep as notation for the automorphism group the same as in Remark 2.3.1. Since the radical of G_0 is solvable and acts on the projective variety X, it has a fixed point: being normal in G_0, it is trivial. Analogously, the center of G_0 - which is contained in a maximal torus - is trivial. Moreover, the hypothesis $\mathrm{Pic}\, X = \mathbf{Z}$ together with Theorem 5.1.9 imply that G_0 is simple. So the group G_0 is simple adjoint and uniquely determined by the variety X, while P_0 is a parabolic subgroup whose reduced subgroup is maximal. Its conjugacy class is uniquely determined by X up to an automorphism of the Dynkin diagram of G_0. Moreover, since the action of G_0 on X is faithful, by Theorem 3.1.2 we have that P_0 is reduced, hence of the form $P_0 = P^\alpha$ for a simple root α.

Now, let us consider the action of G on X: we want to relate in all possible cases the pair (G, P) to the pair (G_0, P_0). This will give us a way to determine, given two homogeneous spaces G/P and G'/P', whether they are isomorphic as varieties. Let us recall that, as in Proposition 2.5.12, we denote as $\overline{G}$ the simple, simply connected group whose root system is dual to the root system of G. The group $\overline{G}$ is the target of the very special isogeny of G.

Proposition 3.3.4. *If the pair (G_0, P_0) is not exceptional in the sense of Demazure, then one of the following two cases holds :*

- *(a) $G = G_0$ and $P = {}_mGP^\alpha$, where $P^\alpha = P_0$ up to an automorphism of the Dynkin diagram of G,*
- *(b) $G = (\overline{G_0})_{ad}$ and $P = {}_mN_GP^\alpha$, where $P^\alpha = \pi_{G_0}(P_0)/Z(\overline{G_0})$ up to an automorphism of the Dynkin diagram of G.*

If (G_0, P_0) is exceptional, then there are two additional possibilities - denoting as (G_0', P_0') the associated pair in the sense of Demazure :

(a') $G = G_0'$ *and* $P = {}_mGP^\alpha$, *where* $P^\alpha = P_0'$ *up to an automorphism of the Dynkin diagram of* G,

(b') $G = (\overline{G_0'})_{ad}$ *and* $P = {}_mN_GP^\alpha$, *where* $P^\alpha = \pi_{G_0'}(P_0')/Z(\overline{G_0'})$ *up to an automorphism of the Dynkin diagram of* G.

Proof. Let us start by assuming that (G_0, P_0) is not exceptional in the sense of Demazure. By Corollary 3.3.3, either $P = {}_mGP^\alpha$ or $P = {}_mN_GP^\alpha$ for some α. In the first case,

$$X = G/{}_mGP^\alpha = G^{(m)}/(P^\alpha)^{(m)} \simeq G/P^\alpha$$

as varieties, hence by Theorem 3.2.3 this implies $G = \underline{\mathrm{Aut}}^0_X = G_0$ and $P^\alpha = P^0$, leading to (a). In the second case,

$$X = G/{}_mN_GP^\alpha = \overline{G}^{(m)}/(P^{\overline{\alpha}})^{(m)} \simeq \overline{G}/P^{\overline{\alpha}} = \overline{G}_{\mathrm{ad}}/\left(P^{\overline{\alpha}}/Z(\overline{G})\right)$$

as varieties, hence by Theorem 3.2.3 again $\overline{G}_{\mathrm{ad}} = \underline{\mathrm{Aut}}^0_X = G_0$ and $P_0 = P^{\overline{\alpha}}/Z(\overline{G})$. Considering their respective images by the very special isogeny of $\overline{G}_{\mathrm{ad}}$ gives (b).

If (G_0, P_0) is exceptional in the sense of Demazure, Theorem 3.2.3 allows for two additional cases: to get the conclusion it is enough to repeat the same reasoning by replacing (G_0, P_0) with (G_0', P_0'). □

CHAPTER 4

Classification of all parabolic subgroups

Abstract. This chapter brings to an end the classification of non-reduced parabolic subgroups in positive characteristic, especially two and three: they are all obtained as intersections of parabolics having maximal reduced part.

4.1. The statement

We prove the following classification result.

Theorem 4.1.1. *Let P be a parabolic subgroup of a semisimple group G over an algebraically closed field of any characteristic, with reduced part P_I. Then the inclusion*

$$P \subseteq \bigcap_{\alpha \in \Delta \setminus I} \langle P, P^\alpha \rangle \tag{4.1.1}$$

is an equality; in particular, P is intersection of parabolic subgroups with maximal reduced part.

4.1.1. Isogenies. Let us recall a factorisation property of isogenies with simple, simply connected source (proved in Proposition 2.5.12). Then, let us introduce the notion of parabolic subgroup of *quasi-standard* type, slightly generalising the definition of parabolic of standard type (first introduced in Definition 2.3.18) in order to work with small characteristics.

Proposition 4.1.2. *Let G be simple, simply connected and let us consider an isogeny*

$$\xi \colon G \to G'.$$

Then there exists a unique factorisation of ξ:

- *either as the composition of the very special isogeny, followed by an iterated Frobenius morphism and then a central isogeny (which can only occur when the edge hypothesis is satisfied by G);*
- *or as the composition of an iterated Frobenius morphism with a central isogeny.*

Definition 4.1.3. A parabolic subgroup is said to be *of quasi-standard type* if it is of the form

$$P = \bigcap_{\alpha \in \Delta \setminus I} (\ker \xi_\alpha) P^\alpha$$

for some isogenies ξ_α with no central factor.

Let us notice that by Proposition 4.1.2, the subgroup $\ker \xi_\alpha$ is necessarily of the form ${}_mGP^\alpha$ or ${}_mN_GP^\alpha$. Moreover, the latter are enough to describe all parabolic subgroups having maximal reduced part, in all types except for a group of type G_2 in characteristic 2, in view of the following classification result, proved in Theorem 3.3.2.

Proposition 4.1.4. *Let P be a parabolic subgroup of G such that its reduced subgroup is equal to P^α for some simple root α.*
Then there exists a unique isogeny ξ with no central factor such that

$$P = (\ker \xi)P^\alpha,$$

unless G is of type G_2 in characteristic 2 and α is the short simple root.

Remark 4.1.5. With respect to the order given in Remark 2.5.13, ξ is *minimal*: in other words, for any isogeny ζ such that $\ker \zeta$ is strictly contained in $\ker \xi$, the subgroup P is not contained in $(\ker \zeta)P^\alpha$. This is a crucial point in the proof of Theorem 4.1.1.

Let us consider a semisimple, simply connected group G, together with a Borel subgroup B and a maximal torus T, and keep all the previous notation. The first step is the following result, which allows us to reduce to the case of a simple group.

Lemma 4.1.6. *Any parabolic subgroup P of a semisimple simply connected group*

$$G = G_1 \times \cdots \times G_n$$

is a product of the parabolic subgroups $P_i := P \cap G_i$ of the simple factors G_i.

In particular, from now on we can (and will) assume the group G to be simple.

Proof. Let us start by the following : we have

$$P = U_P^- \cdot P_{\mathrm{red}}, \quad U_P^- \cap P_{\mathrm{red}} = 1,$$

where U_P^- denotes the intersection of P with the unipotent radical of the opposite of P_{red}; this structure result has been recalled in (2.3.2). Thus, we can see P as a product of its unipotent infinitesimal part and its reduced part. Moreover, we have the following isomorphism:

$$U_P^- = \prod_{\gamma \in \Phi^+ \setminus \Phi_I} (U_P^- \cap U_{-\gamma}),$$

where $I \subset \Delta$ is the basis for the root system of a Levi subgroup. This implies that

$$U_P^- = U_{P_1}^- \times \cdots \times U_{P_n}^-.$$

On the other hand, it is a classical fact that

$$P_{\mathrm{red}} = (P_1)_{\mathrm{red}} \times \cdots \times (P_n)_{\mathrm{red}}.$$

Putting these equalities together allows to conclude that P is the product of the P_is. ☐

4.1.2. Divisors and contractions. Consider a homogeneous variety

$$X = G/P, \quad \text{with } P_{\mathrm{red}} = P_I$$

and denote as o its base point. As explained in detail in Section 5.1, there is a canonical basis of the Picard group of X, given by the so-called *Schubert divisors*. The latter are associated to simple roots and defined as

$$D_\alpha := \overline{B^- s_\alpha o}, \quad \alpha \in \Delta \setminus I, \tag{4.1.2}$$

where B^- denotes the Borel subgroup opposite to B. In Section 5.2, we build a finite family of morphisms

$$f_\alpha \colon X \longrightarrow G/Q^\alpha := \operatorname{Proj} \bigoplus_{m \geq 0} H^0(X, \mathcal{O}_X(mD_\alpha)), \quad \alpha \in \Delta \backslash I. \tag{4.1.3}$$

We construct f_α as the the unique contraction (see Lemma 5.2.4 for the general construction) on G/P such that the Schubert curves

$$C_\beta := \overline{U_{-\beta} o}, \quad \beta \in \Delta \backslash I,$$

which are smooth, are all contracted to a point except for C_α. By Blanchard's Lemma (see Theorem 5.2.3 below), there is a unique G-action on the target making the morphism f_α equivariant: what we denote by Q^α is the stabilizer of such an action. Then we show that Q^α is the subgroup generated by P and P^α.

Lemma 4.1.7. *In all types except for G_2 in characteristic 2, there is a unique isogeny ξ_α with no central factor, such that*

$$Q^\alpha = \langle P, P^\alpha \rangle = (\ker \xi_\alpha) P^\alpha.$$

Proof. By construction, Q^α has maximal reduced part equal to P^α, hence we can apply Proposition 4.1.4 and Remark 4.1.5 and we are done. □

4.2. Height on simple root subgroups

Let us start by proving a Lemma which is repeatedly used in the proof of Theorem 4.1.1. This result tells us that the inclusion (4.1.1) is *not so far* from being an equality: more precisely, when intersecting with the root subgroup associated to the opposite of a simple root, one gets the same height on both sides.

Lemma 4.2.1. *Let P be any parabolic subgroup of G and α a simple root. Then*

$$U_{-\alpha} \cap P = U_{-\alpha} \cap Q^\alpha.$$

Proof. The natural G-equivariant morphism

$$g_\alpha \colon X = G/P \longrightarrow \mathbf{P}(H^0(X, \mathcal{O}_X(D_\alpha))^\vee)$$

is well-defined since D_α is globally generated (see Theorem 5.1.9 below). The canonical section of $H^0(X, \mathcal{O}_X(D_\alpha))$, corresponding to a hyperplane H_α, gives the equality

$$g_\alpha^* \mathcal{O}(H_\alpha) = \mathcal{O}_X(D_\alpha). \tag{4.2.1}$$

On the other hand, the inclusion of the k-subalgebra generated by elements of degree one into the direct sum $\oplus_{m \geq 0} H^0(X, \mathcal{O}_X(mD_\alpha))$ gives a finite morphism h_α, making the following diagram commute.

$$\begin{array}{ccc} X = G/P & \xrightarrow{\quad g_\alpha \quad} & \mathbf{P}(H^0(X, \mathcal{O}_X(D_\alpha))^\vee) \\ \big\downarrow f_\alpha & \nearrow h_\alpha & \\ \operatorname{Proj} \bigoplus_{m \geq 0} H^0(X, \mathcal{O}_X(mD_\alpha)) = G/Q^\alpha & & \end{array}$$

Since f_α is a contraction and h_α is finite, the above diagram is the Stein factorisation of the morphism g_α. Let us denote as E_α and S_α respectively the Schubert divisor and the Schubert curve in G/Q^α. Then set-theoretically the pre-image of E_α is D_α, while the image of C_α is S_α. This means that there are some positive integers m_α and n_α such that $f_\alpha^* E_\alpha = m_\alpha D_\alpha$ and $(f_\alpha)_* C_\alpha = n_\alpha S_\alpha$. The equality (4.2.1) yields

$$D_\alpha = g_\alpha^* H_\alpha = f_\alpha^* h_\alpha^* H_\alpha,$$

hence m_α must be equal to 1. Moreover,

$$1 = D_\alpha \cdot C_\alpha = f_\alpha^* E_\alpha \cdot C_\alpha = E_\alpha \cdot (f_\alpha)_* C_\alpha = n_\alpha E_\alpha \cdot S_\alpha = n_\alpha.$$

This means that f_α restricts to an isomorphism from C_α to S_α: considering the restriction to the respective affine open cells, we get

$$U_{-\alpha}/(U_{-\alpha} \cap P) = U_{-\alpha}/(U_{-\alpha} \cap Q^\alpha)$$

as wanted. □

4.3. Proof of the main result

This section is dedicated to the proof of Theorem 4.1.1. By Lemma 4.1.6, we may and will assume the group G to be simple.

4.3.1. Ingredients for the proof. We use the classification results proven in Theorem 3.3.2 and Proposition 3.2.26, as well as Lemma 4.2.1 and the factorisation of isogenies with simply connected source. Before moving on to the proof, which is a case-by-case argument, let us fix the notation and state a few remarks of which we repeatedly make use below.

As in Definition 2.3.15, we associate to P the numerical function

$$\varphi \colon \Phi \longrightarrow \mathbf{N} \cup \{\infty\}$$

defined as follows: for a root γ, the integer $\varphi(\gamma)$ is the height of the intersection

$$U_{-\gamma} \cap P$$

when γ is positive and not in the Levi subgroup of P_I; and we extend it to $\varphi(\gamma) = \infty$ otherwise. Analogously, φ_i denotes the associated function to $Q^i := Q^{\alpha_i}$, where we keep the notation of [**Bou**] concerning the standard bases of root systems.

Remark 4.3.1. (a) : In particular, we can reformulate Lemma 4.2.1 as

$$\varphi(\alpha_i) = \varphi_i(\alpha_i) = \mathrm{ht}((\ker \xi_i) \cap U_{-\alpha_i}).$$

(b) : Let us recall that two parabolic subgroups P and P', with respective associated functions φ and φ' and with same reduced part P_I, coincide if and only if

$$P \cap U_{-\gamma} = P' \cap U_{-\gamma} \quad \text{for all } \gamma \in \Phi^+ \backslash \Phi_I,$$

which is equivalent to saying that φ and φ' are equal. This is proven in [**Wen**, Proposition 8].

(c) : The functions φ_i always coincide on roots of the same length. More precisely, if the isogeny ξ_i is an iterated Frobenius, then φ_i is constant on all positive roots not in Φ_I. On the other hand, if ξ_i is the composition of an m-th iterated Frobenius morphism and a very special isogeny, then

$$m + 1 = \varphi_i(\gamma) = \varphi_i(\delta) + 1$$

for all short roots γ and all long roots δ not belonging to Φ_I. The two integers above are invariant under the action of the Weyl group, which has one orbit if G is simply laced and two orbits (of long and short roots respectively) otherwise.

Let us recall the following result on structure constants: see [**Hum**, Chapter VII, 25.2].

Lemma 4.3.2. *Let $\gamma \neq \pm\delta$ be roots and r a natural number such that*

$$\gamma - r\delta, \ldots, \gamma, \gamma + \delta$$

are roots but $\gamma - (r+1)\delta$ is not. Then the corresponding vectors of the Chevalley basis satisfy

$$[X_\gamma, X_\delta] = \pm(r+1)X_{\gamma+\delta}.$$

The integer r only depends on the roots γ and δ, and it is uniquely defined because we are working with a fixed Chevalley basis. We denote it as

$$\mathcal{N}(\gamma, \delta)$$

in what follows. This allows us to formulate the following slight adaptation of an argument in [**Wen**].

Lemma 4.3.3. *Let γ and δ be roots such that $\gamma + \delta$ is a root but $\gamma - \delta$ is not. Then*

$$\varphi(\gamma + \delta) \geq \min\{\varphi(\gamma), \varphi(\delta)\}.$$

Proof. Let $m := \varphi(\gamma + \delta)$ be the height of

$$U_{-\gamma-\delta} \cap P.$$

Let us consider $a, b \in \mathbf{G}_a$ such that $u_{-\gamma}(a)$ and $u_{-\delta}(b)$ are in P. Then the commutator

$$(u_{-\gamma}(a), u_{-\delta}(b)) = \prod u_{-i\gamma-j\delta}(c_{ij}a^i b^j),$$

where the product ranges over the finite set of couples of positive integers (i, j) such that $i\gamma + j\delta$ is a root, has a factor $u_{-\gamma-\delta}(\pm ab)$, because

$$c_{11} = \mathcal{N}(\gamma, \delta) = \pm 1.$$

By [**Wen**, Proposition 8], $u_{-\gamma-\delta}(ab)$ belongs to $U_{-\gamma-\delta} \cap P$, hence $(ab)^{p^m}$ vanishes. In particular either a^{p^m} or b^{p^m} is zero, which means that the minimum between the height of $U_{-\gamma} \cap P$ and $U_{-\delta} \cap P$ is less than or equal to m, as wanted. □

4.3.2. Type B_n and C_n. In this section we consider a simply connected group G of type B_n or C_n, over an algebraically closed field of characteristic $p = 2$. Let

- $\overline{P}$ be the pull-back of P via the very special isogeny
$$\overline{\pi} := \pi_{\overline{G}} \colon \overline{G} \longrightarrow G;$$
- $\alpha_i \leftrightarrow \overline{\alpha}_i$ the bijection on simple roots induced by $\overline{\pi}$, which we recall exchanges long and short roots;
- ψ and ψ_i the respective associated functions to $\overline{P}$ and to
$$\ker(\xi_i \circ \overline{\pi}) P^{\overline{\alpha}_i} = \langle Q, P^{\overline{\alpha}_i} \rangle.$$

The situation is summarized in the following diagram.

$$\begin{array}{ccccc} \overline{P} & \hookrightarrow & \bigcap_{\alpha_i \in \Delta \setminus I} \ker(\xi_i \circ \overline{\pi}) P^{\overline{\alpha}_i} & \hookrightarrow & \overline{G} \\ \downarrow & & & & \downarrow \overline{\pi} \\ P & \hookrightarrow & \bigcap_{\alpha_i \in \Delta \setminus I} (\ker \xi_i) P^{\alpha_i} & \hookrightarrow & G \end{array}$$

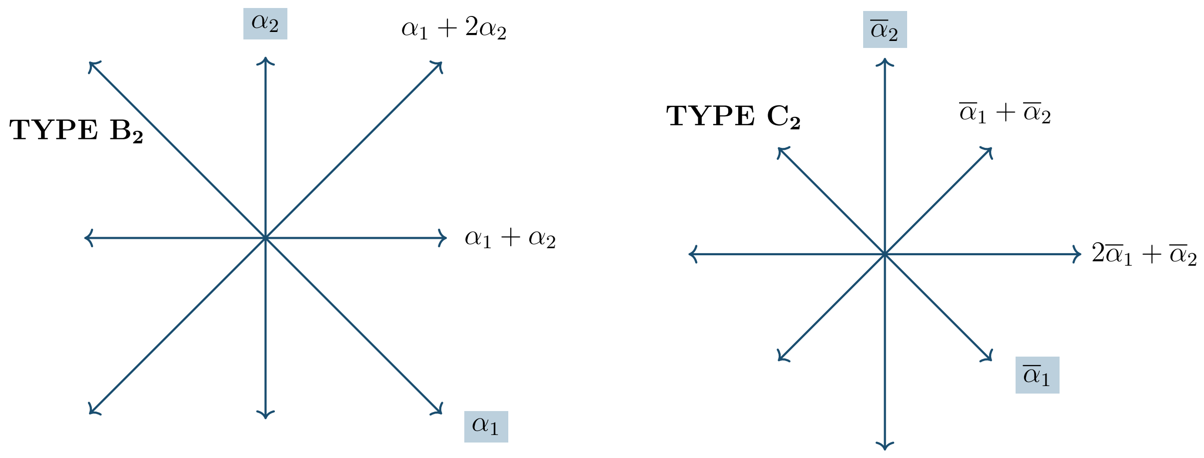
Lemma 4.3.4. *If (4.1.1) is an equality for all parabolic subgroups of a group of type B_n, then the same holds in type C_n.*

Proof. We make use of the above diagram: let G be of type C_n, then by assumption (4.1.1) holds for the parabolic subgroup $\overline{P}$, which means that
$$\psi(\overline{\gamma}) = \min_i \{\psi_i(\overline{\gamma})\}$$
for all positive roots γ of G, where $\gamma \leftrightarrow \overline{\gamma}$ is the bijection induced by the very special isogeny. This implies
$$\begin{aligned} \varphi(\gamma) &= \psi(\overline{\gamma}) = \min_i \{\psi_i(\overline{\gamma})\} = \min_i \{\varphi_i(\gamma)\} && \text{if } \gamma \text{ is short,} \\ \varphi(\gamma) &= \psi(\overline{\gamma}) - 1 = \min_i \{\psi_i(\overline{\gamma}) - 1\} = \min_i \{\varphi_i(\gamma)\} && \text{if } \gamma \text{ is long,} \end{aligned}$$
so we are done. □

In order to prove Theorem 4.1.1 for a group of type B_n, we proceed by induction on n, of which the first step is the case $n = 2$ below.

Lemma 4.3.5. *Let*
$$P \subset (\ker \xi_1) P^{\alpha_1} \cap (\ker \xi_2) P^{\alpha_2} \subset \mathrm{Spin}_5$$
be a parabolic subgroup in type B_2, with reduced part the Borel subgroup.
Then (4.1.1) is an equality.

Proof. Below is a picture of the respective root systems (which are isomorphic) in order to visualise the very special isogeny and make the proof easier to read: a basis in type B_2 is given by a long root α_1 and a short root α_2, while in type C_2 it is given by a short root $\overline{\alpha}_1$ and a long root $\overline{\alpha}_2$.



By Remark 4.3.1 (b), it is enough to prove the following:

$$\varphi(\alpha_1+\alpha_2) \geq \min\{\varphi_1(\alpha_1+\alpha_2), \varphi_2(\alpha_1+\alpha_2)\}; \tag{4.3.1}$$

$$\varphi(\alpha_1+2\alpha_2) \geq \min\{\varphi_1(\alpha_1+2\alpha_2), \varphi_2(\alpha_1+2\alpha_2)\}, \tag{4.3.2}$$

because the opposite inequalities are already implied by the inclusion (4.1.1).
Let us consider the two integers

$$r_1 := \varphi_1(\alpha_1) \quad \text{and} \quad r_2 := \varphi_2(\alpha_2).$$

Given that α_1 is a long root, we have either

$$\xi_1 = F^{r_1} \quad \text{or} \quad \xi_1 = F^{r_1} \circ \pi, \tag{4.3.3}$$

because by Proposition 4.1.2 these are the only two isogenies with no central factor whose kernel has height r_1 on long root subgroups. Analogously, α_2 being a short root, we have either

$$\xi_2 = F^{r_2} \quad \text{or} \quad \xi_2 = F^{r_2-1} \circ \pi. \tag{4.3.4}$$

for the same reason.
Step 1: Let us start by considering the root $\alpha_1+\alpha_2$. We have that $\mathcal{N}(\alpha_1,\alpha_2) = \pm 1$, so

$$\begin{aligned} \varphi(\alpha_1+\alpha_2) &\geq \min\{\varphi(\alpha_1), \varphi(\alpha_2)\} && \text{by Lemma 4.3.3,} \\ &= \min\{\varphi_1(\alpha_1), \varphi_2(\alpha_2)\} && \text{by Remark 4.3.1 (a).} \end{aligned}$$

This translates into the inequality

$$\varphi(\alpha_1+\alpha_2) \geq \min\{r_1, r_2\}. \tag{4.3.5}$$

Since α_2 and $\alpha_1+\alpha_2$ are both short, we have

$$\varphi_2(\alpha_1+\alpha_2) = r_2$$

by Remark 4.3.1(c). Moreover, if ξ_1 is an r_1-th iterated Frobenius, then

$$\varphi_1(\alpha_1+\alpha_2) = r_1$$

so that by (4.3.5) we are done. So let us assume that

$$\xi_1 = F^{r_1} \circ \pi, \tag{4.3.6}$$

which in particular means

$$\varphi_1(\alpha_1+\alpha_2) = r_1+1.$$

What is left to prove in this case is that

$$\varphi(\alpha_1+\alpha_2) \geq \min\{r_1+1, r_2\}.$$

Now, if $r_2 \leq r_1$, then (4.3.5) becomes

$$\varphi(\alpha_1+\alpha_2) \geq \min\{r_1, r_2\} = r_2 = \min\{r_1+1, r_2\}$$

and we are done. So let us assume that $r_1 < r_2$: again by (4.3.5), it is enough to get a contradiction with the assumption

$$\varphi(\alpha_1+\alpha_2) = r_1.$$

Let us assume this last equality to be true, and remark that by (4.3.4), since $\alpha_1+2\alpha_2$ is a long root we have

$$\varphi_2(\alpha_1+2\alpha_2) \leq r_2$$

in both cases. Next, the inclusion (4.1.1) gives

$$\varphi(\alpha_1+2\alpha_2) \leq \min\{\varphi_1(\alpha_1+2\alpha_2), \varphi_2(\alpha_1+2\alpha_2)\} \leq \min\{r_1, r_2\} = r_1.$$

In particular, all positive roots γ whose support contains α_1 satisfy $\varphi(\gamma) \leq r_1$. In other words, the subgroup P is contained in ${}_{r_1}GP^{\alpha_1}$. However, this implies that

$$\langle P, P^{\alpha_1}\rangle = (\ker\xi_1)P^{\alpha_1} \subset {}_{r_1}GP^{\alpha_1},$$

which contradicts (4.3.6).

Step 2: Let us move on to the root $\alpha_1+2\alpha_2$: consider the pull-back

$$\overline{P} := \overline{\pi}^{-1}(P) \subseteq \ker(\xi_1 \circ \overline{\pi})P^{\overline{\alpha}_1} \cap \ker(\xi_2 \circ \overline{\pi})P^{\overline{\alpha}_2},$$

which is a parabolic subgroup of Sp_4.
The very special isogeny $\overline{\pi}$ sends $\overline{\alpha}_1+\overline{\alpha}_2$ to $\alpha_1+2\alpha_2$: in particular, pulling back via $\overline{\pi}$ gives

$$\psi(\overline{\alpha}_1+\overline{\alpha}_2) = \varphi(\alpha_1+2\alpha_2)+1 \tag{4.3.7}$$

and the analogous equalities hold for ψ_1 and ψ_2. Thus proving (4.3.2) is equivalent to showing that

$$\psi(\overline{\alpha}_1+\overline{\alpha}_2) \geq \min\{\psi_1(\overline{\alpha}_1+\overline{\alpha}_2), \psi_2(\overline{\alpha}_1+\overline{\alpha}_2)\}. \tag{4.3.8}$$

Now, the structure constant $\mathcal{N}(\overline{\alpha}_1, \overline{\alpha}_2)$ is equal to ± 1, so we have

$$\begin{aligned} \psi(\overline{\alpha}_1+\overline{\alpha}_2) &\geq \min\{\psi(\overline{\alpha}_1), \psi(\overline{\alpha}_2)\} && \text{by Lemma 4.3.3} \\ &= \min\{\psi_1(\overline{\alpha}_1), \psi_2(\overline{\alpha}_2)\} && \text{by Remark 4.3.1 (a).} \end{aligned}$$

This translates into the inequality

$$\psi(\overline{\alpha}_1+\overline{\alpha}_2) \geq \min\{r_1+1, r_2\}. \tag{4.3.9}$$

Next, $\overline{\alpha}_1+\overline{\alpha}_2$ and $\overline{\alpha}_1$ are both short, which implies

$$\psi_1(\overline{\alpha}_1+\overline{\alpha}_2) = \psi_1(\overline{\alpha}_1) = r_1+1$$

by Remark 4.3.1(c). On the other hand, $\overline{\alpha}_2$ is long, hence (again by Proposition 4.1.2)

$$\psi_2(\overline{\alpha}_1+\overline{\alpha}_2)=\begin{cases}\psi_2(\overline{\alpha}_2)=r_2 & \text{if } \xi_2\circ\overline{\pi}=F^{r_2};\\ \psi_2(\overline{\alpha}_2)+1=r_2+1 & \text{if } \xi_2=F^{r_2}.\end{cases}$$

In the first case, (4.3.8) automatically holds, so let us assume we are in the second one, namely that the isogeny ξ_2 is an r_2-th iterated Frobenius, which in particular means that

$$\psi_2(\overline{\alpha}_1+\overline{\alpha}_2)=r_2+1.$$

What is left to prove in this case is that

$$\psi(\overline{\alpha}_1+\overline{\alpha}_2)\geq\min\{r_1+1,r_2+1\}.$$

Now, if $r_2>r_1$, then (4.3.9) becomes

$$\psi(\overline{\alpha}_1+\overline{\alpha}_2)\geq\min\{r_1+1,r_2\}=r_1+1=\min\{r_1+1,r_2+1\}$$

and we are done. So let us assume that $r_2\leq r_1$. Again by (4.3.9), it is enough to get a contradiction with the assumption

$$\psi(\overline{\alpha}_1+\overline{\alpha}_2)=r_2.$$

If this last assumption holds, then pushing forward to P and using (4.3.7) gives

$$\varphi(\alpha_1+2\alpha_2)=\psi(\overline{\alpha}_1+\overline{\alpha}_2)-1=r_2-1.$$

Putting this together with (4.3.1), which has been proved in Step 1 above, yields

$$\varphi(\alpha_1+\alpha_2)=\min\{\varphi_1(\alpha_1+\alpha_2),\varphi_2(\alpha_1+\alpha_2)\}\leq r_2.$$

In particular, the subgroup P is contained in $\ker(F^{r_2-1}\circ\overline{\pi})P^{\alpha_2}$, thus implying

$$\langle P,P^{\alpha_2}\rangle=(\ker\xi_2)P^{\alpha_1}\subset\ker(F^{r_2-1}\circ\overline{\pi})P^{\alpha_2},$$

which contradicts the assumption that $\xi_2=F^{r_2}$. □

Proposition 4.3.6. *Let P be as in the diagram above in type B_n or C_n. Then (4.1.1) is an equality.*

Proof. Lemma 4.3.4 and Lemma 4.3.5 imply that the statement holds for $n=2$. Let us assume it to be true in types B_n and C_n, with respective basis

$$\alpha_1,\ \ldots,\ \alpha_n\quad\text{and}\quad\overline{\alpha}_1,\ \ldots,\ \overline{\alpha}_n,$$

and consider the groups of type B_{n+1} and C_{n+1}, with respective basis

$$\alpha_0,\ \alpha_1,\ \ldots,\ \alpha_n\quad\text{and}\quad\overline{\alpha}_0,\ \overline{\alpha}_1,\ \ldots,\ \overline{\alpha}_n.$$

First, by Lemma 4.3.4 we can reduce to the case of a group of type B_{n+1}. Moreover, by Remark 4.3.1(b), it suffices to show that

$$\varphi(\gamma)=\min\{\varphi_i(\gamma)\colon \alpha_i\in\operatorname{Supp}(\gamma)\}\quad\text{for all }\gamma\in\Phi^+\backslash\Phi_I^+. \tag{4.3.10}$$

Step 1: Let us assume that the simple root α_0 is not in the support of γ and let L^{α_0} be the Levi subgroup associated to α_0 (namely, with basis all simple roots except for the first one): then L^{α_0} is of type B_n and the inclusion

$$U_{-\gamma}\subset L^{\alpha_0}$$

is satisfied. We can thus apply the induction hypothesis and conclude that (4.3.10) holds for γ.

Step 2: Let us consider a root of the form

$$\gamma = \varepsilon_0 - \varepsilon_i, \quad 1 \leq i \leq n.$$

Denoting as L^{α_n} the Levi subgroup associated to the last simple root α_n, we see that L^{α_n} is of type A_{n-1} hence simply laced; moreover, the inclusion

$$U_{-\gamma} \subset L^{\alpha_n}$$

is satisfied. In particular, the intersection $P \cap L^{\alpha_n}$ is a parabolic subgroup of L^{α_n}, hence of standard type, from which we deduce the equality (4.3.10) for γ.

Step 3: Next, let us consider the root

$$\gamma = \varepsilon_0.$$

The structure constant $\mathcal{N}(\varepsilon_0 - \varepsilon_1, \varepsilon_1)$ is equal to ± 1, so we have

$$\begin{aligned} \varphi(\varepsilon_0) &\geq \min\{\varphi(\varepsilon_0 - \varepsilon_1), \varphi(\varepsilon_1)\} && \text{by Lemma 4.3.3} \\ &= \min\{\varphi_0(\alpha_0), \varphi(\varepsilon_1)\} && \text{by Remark 4.3.1 (a)} \\ &= \min\{\varphi_0(\alpha_0), \varphi_i(\varepsilon_1),\ i > 0\} && \text{by Step 1} \\ &= \min\{\varphi_0(\alpha_0), \varphi_i(\varepsilon_0),\ i > 0\} && \text{by Remark 4.3.1(c).} \end{aligned}$$

If α_0 belongs to I then we are done because $\varphi_0(\alpha_0) = \infty$. Hence we can assume α_0 to be in $\Delta \backslash I$ and set

$$r_0 := \varphi_0(\alpha_0) \quad \text{and} \quad r := \min\{\varphi_i(\varepsilon_0),\ i > 0\}.$$

Since α_0 is a long root, by Proposition 4.1.2 the isogeny ξ_0 is either equal to an r_0-th iterated Frobenius or to its composition with a very special isogeny. In the first case we directly have (4.3.10) and we are done, so let us assume that

$$\xi_0 = F^{r_0} \circ \overline{\pi}. \tag{4.3.11}$$

In particular, $\varphi_0(\varepsilon_0)$ is equal to $m + 1$. Thus, the inequalities above become

$$\varphi(\varepsilon_0) \geq \min\{r_0, r\}.$$

If r is less than or equal to r_0 we are again done, so let us assume $r_0 < r$. We want to exclude the possibility of a strict inequality i.e.

$$r_0 = \varphi(\varepsilon_0).$$

However, since ε_0 is the only short root containing α_0 in its support, this last assumption implies that P is contained in ${}_{r_0}GP^{\alpha_0}$. In particular,

$$\langle P, P^{\alpha_0} \rangle = (\ker \xi_0) P^{\alpha_0} \subset {}_m GP^{\alpha_0},$$

which contradicts (4.3.11). This allows to conclude that (4.3.10) is always true for the root ε_0.

Step 4: We have left to prove (4.3.10) for the roots

$$\gamma = \varepsilon_0 + \varepsilon_i, \quad 1 \leq i \leq n.$$

In this case, the support is equal to the whole of Δ. If $i \neq 1$, we can use the fact that the structure constant $\mathcal{N}(\varepsilon_0 - \varepsilon_1, \varepsilon_1 + \varepsilon_i)$ is equal to ± 1 to obtain

$$\begin{aligned}
\varphi(\varepsilon_0 + \varepsilon_i) &\geq \min\{\varphi(\varepsilon_0 - \varepsilon_1), \varphi(\varepsilon_1 + \varepsilon_i)\} && \text{by Lemma 4.3.3}\\
&= \min\{\varphi_0(\alpha_0), \varphi(\varepsilon_1 + \varepsilon_i)\} && \text{by Remark 4.3.1 (a)}\\
&= \min\{\varphi_0(\alpha_0), \varphi_l(\varepsilon_1 + \varepsilon_i),\ l > 0\} && \text{because (4.3.10) holds for } \varepsilon_1 + \varepsilon_i\\
&= \min_j\{\varphi_j(\varepsilon_0 + \varepsilon_i)\} && \text{by Remark 4.3.1(c).}
\end{aligned}$$

On the other hand, if $i = 1$, we can proceed analogously using the fact that the structure constant $\mathcal{N}(\varepsilon_0 - \varepsilon_n, \varepsilon_1 + \varepsilon_n)$ is equal to ± 1, and we get

$$\begin{aligned}
\varphi(\varepsilon_0 + \varepsilon_1) &\geq \min\{\varphi(\varepsilon_0 - \varepsilon_n), \varphi(\varepsilon_1 + \varepsilon_n)\} && \text{by Lemma 4.3.3}\\
&= \min\{\varphi_j(\varepsilon_0 - \varepsilon_n),\ j < n,\ \varphi(\varepsilon_1 + \varepsilon_n)\} && \text{because (4.3.10) holds for } \varepsilon_0 - \varepsilon_n\\
&= \min\{\varphi_j(\varepsilon_0 - \varepsilon_n),\ j < n,\ \varphi_k(\varepsilon_1 + \varepsilon_n),\ k > 0\} && \text{because (4.3.10) holds for } \varepsilon_1 + \varepsilon_n\\
&= \min_j\{\varphi_j(\varepsilon_0 + \varepsilon_1)\} && \text{by Remark 4.3.1(c).}
\end{aligned}$$

This concludes the proof. □

Lemma 4.3.7. *Let us assume that we are working over a field of characteristic $p = 3$. Then any parabolic subgroup of a simple group of type B_n or C_n is standard. In particular, Theorem 4.1.1 also holds in this case.*

We follow the reasoning in [**Wen**, Theorem 10], and we use the fact that all structure constants appearing in the proof have absolute value strictly smaller than 3.

PROOF. Keeping the above notation, the inclusion (4.1.1) becomes

$$P \subseteq \bigcap_{\alpha \in \Delta \setminus I} {}_{m_\alpha}GP^\alpha,$$

where, by Remark 4.3.1(a) we have

$$m_\alpha = \varphi(\alpha) = \varphi_\alpha(\alpha).$$

It is thus enough to prove the inequality

$$\varphi(\gamma) \geq \min\{m_\alpha \colon \alpha \in \operatorname{Supp}(\gamma)\},$$

by induction on the height of γ. If γ is simple then the statement is true. If not, let $\alpha \in \operatorname{Supp}(\gamma)$ such that $\gamma - \alpha$ is still a root. Then the structure constant

$$\mathcal{N}(\gamma - \alpha, \alpha)$$

is equal to ± 1 or to ± 2; in particular it never vanishes over the base field. This implies, by the same reasoning as the one in the proof of Lemma 4.3.3, that

$$\varphi(\gamma) \geq \min\{\varphi(\gamma - \alpha), \varphi(\alpha)\},$$

and we are done by the induction hypothesis. □

4.3.3. Type F_4. Consider a group G of type F_4 over an algebraically closed field of characteristic $p > 0$ (in particular $p = 2$ or $p = 3$ are the interesting cases for us).
Let us adapt all previous notation:

is the Dynkin diagram, and we denote as

$$Q^i = \langle P, P^{\alpha_i}\rangle = (\ker \xi_i)P^{\alpha_i}, \quad \alpha_i \in \Delta \backslash I,$$

the family of parabolic subgroups with maximal reduced part associated to P. Moreover, we call φ and φ_i the associated functions to P and to $\langle P, P^{\alpha_i}\rangle$ respectively.

Proposition 4.3.8. *The inclusion (4.1.1) is an equality in type F_4.*

Proof. By Remark 4.3.1(b), it is enough to show that

$$\varphi(\gamma) \geq \min_i\{\varphi_i(\gamma)\}, \quad \text{for all } \gamma \in \Phi^+ \backslash \Phi_I^+. \tag{4.3.12}$$

Step 1: Let us assume that the support of γ is not equal to the whole of Δ. Then $U_{-\gamma}$ is contained in a Levi subgroup of the form

$$L := P^{\alpha_i} \cap (P^{\alpha_i})^-$$

for some i. In particular, L has root system of type C_3 (if α_1 is not in $\mathrm{Supp}(\gamma)$), of type B_3 (if α_4 is not in $\mathrm{Supp}(\gamma)$) or of type $A_1 \times A_2$ (if one among α_2 and α_3 is not in $\mathrm{Supp}(\gamma)$). In any of these situations, the inequality (4.3.12) holds for γ, because

$$U_{-\gamma} \cap P = U_{-\gamma} \cap L$$

have same height, and because any parabolic subgroup in type C_3, B_3 and $A_1 \times A_2$ is intersection of parabolic subgroups with maximal reduced part.

Step 2: Let us consider a long root γ whose support is equal to Δ. By Lemma 4.3.9 proven below, the subgroup H generated by long root subgroups is of type D_4. Let us consider the following basis for the root system of H, as in (4.3.18):

$$\beta_1 := \alpha_2 + 2\alpha_3 + \alpha_4 = \varepsilon_1 - \varepsilon_2, \qquad \beta_2 := \alpha_1 = \varepsilon_2 - \varepsilon_3,$$
$$\beta_3 := \alpha_2 = \varepsilon_3 - \varepsilon_4, \qquad \beta_4 := \alpha_2 + 2\alpha_3 = \varepsilon_3 + \varepsilon_4.$$

Since γ is a long root satisfying

$$U_{-\gamma} \cap P = U_{-\gamma} \cap H,$$

we can use the fact that the group of type D_4 is simply laced to get

$$\varphi(\gamma) = \min_i\{\varphi(\beta_i)\}. \tag{4.3.13}$$

Next, let us consider again the β_is as being roots of G; notice that their support is not equal to the whole of Δ. This allows us to apply Step 1 to β_i, to obtain

$$\varphi(\beta_i) = \min_j\{\varphi_j(\beta_i) \colon \alpha_j \in \mathrm{Supp}(\beta_i)\}, \quad 1 \leq i \leq 4.$$

In conclusion, (4.3.13) becomes

$$\varphi(\gamma) = \min_{i,j}\{\varphi_j(\beta_i) \colon \alpha_j \in \mathrm{Supp}(\beta_i)\} = \min_j\{\varphi_j(\gamma)\}$$

where the last equality is due to Remark 4.3.1(c); in particular, (4.3.12) holds for γ.

Step 3: Finally, we are led to consider short roots whose support is equal to the whole of Δ. There are five of them, namely:

$$\begin{aligned}
\delta_1 &= \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = \delta_2 - \alpha_3, \\
\delta_2 &= \alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4 = \delta_3 - \alpha_2, \\
\delta_3 &= \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 = \delta_4 - \alpha_3, \\
\delta_4 &= \alpha_2 + 2\alpha_2 + 3\alpha_3 + \alpha_4 = \delta_5 - \alpha_4, \\
\delta_5 &= \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 = \delta_1 + (\alpha_2 + 2\alpha_3 + \alpha_4).
\end{aligned}$$

Let us recall that we set the function φ to be constant and equal to infinity on negative roots. Moreover, we have

$$\begin{aligned}
&\mathcal{N}(\delta_2, -\alpha_3) = \pm 1, && \text{because } \delta_2 + \alpha_3 \text{ is not a root;} \\
&\mathcal{N}(\delta_3, -\alpha_2) = \pm 1, && \text{because } \delta_3 + \alpha_2 \text{ is not a root;} \\
&\mathcal{N}(\delta_4, -\alpha_3) = \pm 1, && \text{because } \delta_4 + \alpha_3 \text{ is not a root;} \\
&\mathcal{N}(\delta_5, -\alpha_4) = \pm 1, && \text{because } \delta_5 + \alpha_4 \text{ is not a root;} \\
&\mathcal{N}(\delta_1, \nu) = \pm 1, \text{where } \nu = \alpha_2 + 2\alpha_3 + \alpha_4, && \text{because } \delta_1 + \nu \text{ is not a root.}
\end{aligned}$$

The structure constants just above imply, by Lemma 4.3.3, that

$$\varphi(\delta_1) \geq \varphi(\delta_2) \geq \varphi(\delta_3) \geq \varphi(\delta_4) \geq \varphi(\delta_5) \geq \min\{\varphi(\delta_1), \varphi(\nu)\}. \tag{4.3.14}$$

We can now apply Step 1 to the root ν, because its support is not the whole of Δ, as well as Remark 4.3.1(c), to obtain

$$\varphi_i(\nu) = \varphi_i(\delta_1)$$

for all i. In particular,

$$\begin{aligned}
\min\{\varphi(\delta_1), \varphi(\nu)\} &= \min\{\varphi(\delta_1), \varphi_2(\nu), \varphi_3(\nu), \varphi_4(\nu)\} \\
&= \min\{\varphi(\delta_1), \varphi_2(\delta_1), \varphi_3(\delta_1), \varphi_4(\delta_1)\} = \varphi(\delta_1).
\end{aligned}$$

Together with (4.3.14), we can deduce that

$$\varphi(\delta_1) = \varphi(\delta_2) = \varphi(\delta_3) = \varphi(\delta_4) = \varphi(\delta_5),$$

so that it is enough to show

$$\varphi(\delta_1) \geq \min_i\{\varphi_i(\delta_1)\} =: m. \tag{4.3.15}$$

Let us prove (4.3.15): first, let us notice that we have

$$\begin{aligned}
\varphi(\delta_1) &\geq \min\{\varphi(\alpha_1 + \alpha_2 + \alpha_3), \varphi(\alpha_4)\} && \text{because } \mathcal{N}(\alpha_1 + \alpha_2 + \alpha_3, \alpha_4) = \pm 1; \\
&\geq \min\{\varphi(\alpha_1 + \alpha_2), \varphi(\alpha_3), \varphi(\alpha_4)\} && \text{because } \mathcal{N}(\alpha_1 + \alpha_2, \alpha_3) = \pm 1; \\
&\geq \min\{\varphi(\alpha_1), \varphi(\alpha_2), \varphi(\alpha_3), \varphi(\alpha_4)\} && \text{because } \mathcal{N}(\alpha_1, \alpha_2) = \pm 1; \\
&= \min_i\{\varphi_i(\alpha_i)\} && \text{by Remark 4.3.1(a);} \\
&= \min_i\{\varphi_1(\alpha_1), \varphi_2(\alpha_2), \varphi_3(\delta_1), \varphi_4(\delta_1)\} && \text{by Remark 4.3.1(c).}
\end{aligned}$$

If the minimum just above is realised by $\varphi_3(\delta_1)$ or $\varphi_4(\delta_1)$, then (4.3.15) holds and we are done. If the minimum is realised by $\varphi_1(\alpha_1)$ and ξ_1 is an m-th iterated Frobenius, then we have

$$m = \varphi_1(\alpha_1) = \varphi_1(\delta_1)$$

and we are also done; analogously for $\varphi_2(\alpha_2)$.
We are left with the following cases, for which (4.3.15) becomes a strict inequality:

$$(a) : m = \varphi_2(\delta_1) > \varphi_2(\alpha_2) = m-1, \quad \text{i.e. } \xi_2 = F^{m-1} \circ \overline{\pi}, \text{ or}$$
$$(b) : m = \varphi_1(\delta_1) > \varphi_1(\alpha_1) = m-1, \quad \text{i.e. } \xi_1 = F^{m-1} \circ \overline{\pi}.$$

Let us assume we are in one of these two situations. We want to get a contradiction with the assumption (respectively):

$$\text{in } (a),\ m-1 = \varphi(\delta_1) < \varphi_2(\delta_1) = m;$$
$$\text{in } (b),\ m-1 = \varphi(\delta_1) < \varphi_1(\delta_1) = m.$$

We now claim that in both situations (a) and (b) all positive short roots γ containing α_2 in their support satisfy

$$\varphi(\gamma) \leq m-1. \tag{4.3.16}$$

Notice that in the root system of type F_4, all short roots containing α_1 in their support also contain α_2, so that this reasoning works for both (a) and (b).
The inequality (4.3.16) is true for $\delta_1 \dots \delta_5$ by (4.3.14), hence we can consider γ such that its support is not equal to Δ. These roots are

$\gamma_1 = \alpha_1 + \alpha_2 + \alpha_3$, satisfying $\varphi(\gamma_1) \leq \varphi(\delta_1)$ because $\mathcal{N}(\alpha_1 + \alpha_2 + \alpha_3, \alpha_4) = \pm 1$;
$\gamma_2 = \alpha_2 + \alpha_3$, satisfying $\varphi(\gamma_2) \leq \varphi(\gamma_1)$ because $\mathcal{N}(\alpha_2 + \alpha_3, \alpha_1) = \pm 1$;
$\gamma_3 = \alpha_2 + \alpha_3 + \alpha_4$, satisfying $\varphi(\gamma_3) \leq \varphi(\delta_1)$ because $\mathcal{N}(\alpha_2 + \alpha_3 + \alpha_4, \alpha_1) = \pm 1$;
$\gamma_4 = \alpha_1 + 2\alpha_2 + \alpha_3$, satisfying $\varphi(\gamma_4) \leq \varphi(\gamma_3)$ because $\mathcal{N}(\alpha_2 + 2\alpha_3 + \alpha_4, -\alpha_3) = \pm 1$.

In particular, this yields, using Lemma 4.3.3 at each step, that

$$P \subseteq {}_{m-1}GP^{\alpha_2} \quad \text{in case } (a),$$
$$P \subseteq {}_{m-1}GP^{\alpha_1} \quad \text{in case } (b).$$

This would imply that the kernel of ξ_2 (resp. of ξ_1) is contained in ${}_{m-1}G$, which contradicts the assumption (a) (resp. the assumption (b)). Hence (4.3.15) holds and we are done. □

Lemma 4.3.9. *Let G be of type F_4. Then the root subgroups associated to long roots of G generate a semisimple subgroup of type D_4.*

Proof. We proceed in two consecutive steps: the first one realises an embedding of $K = \mathrm{Spin}_9$ into G, while the second one an embedding of $H = \mathrm{Spin}_8$ into K, such that the root system of H consists exactly of all the long roots of G.
Step 1: Keeping the notation for root systems of [Bou], we have that

$$\pm\varepsilon_i, \quad \pm\varepsilon_i \pm \varepsilon_j,\ i \neq j \tag{4.3.17}$$

form a root subsystem of Φ of type B_4, with basis

$$\nu_1 := \varepsilon_1 - \varepsilon_2 = \alpha_2 + 2\alpha_3 + 2\alpha_4, \quad \nu_2 := \varepsilon_2 - \varepsilon_3 = \alpha_1, \quad \nu_3 := \varepsilon_3 - \varepsilon_4 = \alpha_2, \quad \nu_4 := \varepsilon_4 = \alpha_3.$$

The subgroup generated by these roots in the character lattice of the maximal torus T of G is

$$R := \langle \nu_1, \nu_2, \nu_3, \nu_4 \rangle = \langle \alpha_1, \alpha_2, \alpha_3, 2\alpha_4 \rangle \subset X(T)$$

which has index 2. The quotient map

$$X(T) \longrightarrow \mathbf{Z}/2\mathbf{Z}$$

corresponds to an injection $M \subset T$, where M is a copy of μ_2.
Let K be the connected component of the identity of the centralizer of M in G: by Lemma 4.3.10 below, K is smooth and reductive. Its Lie algebra satisfies

$$\operatorname{Lie} K = \operatorname{Lie} C_G(M) = (\operatorname{Lie} G)^M = \operatorname{Lie} T \oplus \{\mathfrak{g}_\gamma : \ tXt^{-1} = X \text{ for all } X \in \mathfrak{g}_\gamma, \, t \in M\}.$$

Let us consider some $t \in M$ and $X \in \mathfrak{g}_\gamma$; by definition of root subspaces, we have

$$tXt^{-1} = \gamma(t)X.$$

Moreover, by construction of M we have $\gamma(t) \in \mu_2$ and $\gamma(M) = 1$ if and only if the coefficient of α_4 (in the unique expression of γ as linear combination of simple roots with integer coefficients of the same sign) is even. This means exactly that $\mathfrak{g}_\gamma$ is contained in $\operatorname{Lie} K$ if and only if γ is one of the roots in (4.3.17). Finally, we can conclude that K is simply connected of type B_4, with the desired set of roots and with maximal torus

$$T' := (T \cap K)^0_{\text{red}}.$$

Step 2: The second step follows by the exact same reasoning, by considering

$$\pm\varepsilon_i \pm \varepsilon_j, \, i \neq j$$

as a root subsystem of type D_4, with basis

$$\beta_1 := \nu_1, \quad \beta_2 := \nu_2, \quad \beta_3 := \nu_3, \quad \beta_4 := \varepsilon_3 + \varepsilon_4 = \nu_3 + 2\nu_4. \tag{4.3.18}$$

These roots generate

$$R' := \langle \beta_1, \beta_2, \beta_3, \beta_4 \rangle = \langle \nu_1, \nu_2, \nu_3, 2\nu_4 \rangle \subset R$$

as a subgroup of index 2. The corresponding quotient map

$$X(T') = R \longrightarrow \mathbf{Z}/2\mathbf{Z}$$

corresponds to an injection $M' \subset T'$, where M' is a copy of μ_2. Thus, we get

$$H := C_K(M')^0$$

as a copy of Spin_8 inside of $\mathrm{Spin}_9 = K \subset G$, having as roots exactly the long roots of $\operatorname{Lie} G$, and we are done. □

Lemma 4.3.10. *Let $M \subset T \subset G$ with G a simply connected semisimple group and T a maximal torus of G.*
Then the identity component of the centralizer $Z := C_G(M)^0$ is smooth and reductive.

Let us mention that Lemma 4.3.10 is a particular case of [CGP, Proposition A.8.12]. We provide a direct, elementary proof below.

Proof. Smoothness is a general fact, since M is linearly reductive.
Next, let us assume that Z is not reductive: since Z is contained in T, there is some root γ of $\operatorname{Lie} G$ such that

$$\mathfrak{g}_\gamma \subset \operatorname{Lie} U, \quad \text{where } U := R_u(Z).$$

Since M is linearly reductive, we have that $\operatorname{Lie} Z$ is the fixed point subalgebra $(\operatorname{Lie} G)^M$. In particular, if the root subspace associated to γ is contained in $\operatorname{Lie} Z$, then the same holds for $-\gamma$. Moreover, U being normal in Z implies that $\operatorname{Lie} U$ is a p-Lie ideal of $\operatorname{Lie} Z$, hence

$$[\mathfrak{g}_\gamma, \mathfrak{g}_{-\gamma}] \subset \operatorname{Lie} U.$$

On the other hand, the above bracket is nonzero and contained in the Lie algebra of the maximal torus T (thanks to the assumption that G is simply connected, we can use a Chevalley basis and apply Lemma 3.1.5). This gives a contradiction. □

4.3.4. Type G_2. Let us consider a group of type G_2: we begin with the case of characteristic 3 because the edge hypothesis is satisfied, and we once again get that all parabolic subgroups are of quasi-standard type. Then we move on to characteristic 2, where a more exotic behavior takes place.

4.3.4.1. *Characteristic three.* Let G be of type G_2 in characteristic 3.

Proposition 4.3.11. *Let $P \subset G$ be a parabolic subgroup with*

$$P_{red} = B = P^{\alpha_1} \cap P^{\alpha_2}.$$

Then P is the pull-back of a parabolic subgroup of standard type *by an isogeny with no central factor; in particular, (4.1.1) is an equality.*

Proof. The first step consists in taking the quotient of P by the kernel of the iterated Frobenius of largest possible height. Hence we can make the hypothesis that the Frobenius kernel ${}_1G$ is not contained in P, which is equivalent to assuming that the Lie algebra of P is not the whole of $\operatorname{Lie} G$. Looking at structure constants, we see that

$$[\mathfrak{g}_{-3\alpha_1-2\alpha_2}, \mathfrak{g}_\gamma] = \mathfrak{g}_{-3\alpha_1-2\alpha_2+\gamma},$$

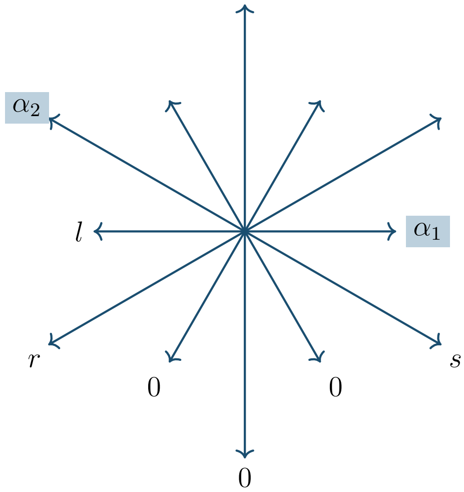
for any positive root γ such that $-3\alpha_1 - 2\alpha_2 + \gamma$ is still a root. Hence, if $\mathfrak{g}_{-3\alpha_1-2\alpha_2}$ intersects $\operatorname{Lie} P$, then all other negative root subspaces do and we get $\operatorname{Lie} P = \operatorname{Lie} G$. Thus by our assumption, we necessarily have

$$\mathfrak{g}_{-3\alpha_1-2\alpha_2} \cap \operatorname{Lie} P = 0. \tag{4.3.19}$$

Moreover, by taking the quotient via the very special isogeny π, which exists because we are in characteristic 3, we can assume that $N := N_G$ is not contained in P. In other words, we make the hypothesis that at least one root subspace associated to a short negative root does not intersect $\operatorname{Lie} P$. Let us notice that

$$[\mathfrak{g}_{-2\alpha_1-\alpha_2}, \mathfrak{g}_{\alpha_1}] = \mathfrak{g}_{-\alpha_1-\alpha_2}, \quad [\mathfrak{g}_{-\alpha_1-\alpha_2}, \mathfrak{g}_{\alpha_2}] = \mathfrak{g}_{-\alpha_1}, \quad [\mathfrak{g}_{-\alpha_1}, \mathfrak{g}_{-\alpha_1-\alpha_2}] = \mathfrak{g}_{-2\alpha_1-\alpha_2},$$

because all three structure constants are equal to ± 2. Thus, if the root subspace associated to $-2\alpha_1$ or to $\alpha_1 - \alpha_2$ intersects $\operatorname{Lie} P$, then $\operatorname{Lie} N$ is contained in $\operatorname{Lie} P$ and we have a contradiction. Hence we are in the following situation, where l, r, s are non-negative integers and we place next to each negative root γ the height of the intersection $P \cap U_\gamma$.



Finally, we can also say that r must be equal to zero, because

$$[\mathfrak{g}_{-3\alpha_1-\alpha_2}, \mathfrak{g}_{\alpha_1}] = \mathfrak{g}_{-2\alpha_1-\alpha_2}.$$

This means that

$$P = {}_l G P^{\alpha_1} \cap {}_s G P^{\alpha_2},$$

with one among l and s which vanishes (because otherwise P would contain ${}_1 G$). ☐

4.3.4.2. *Characteristic two.* Let G be of type G_2 in characteristic 2. Let us briefly recall some results which are shown in Section 3.2. First, there exist two maximal p-Lie subalgebras of $\operatorname{Lie} G$ containing $\operatorname{Lie} P^{\alpha_1}$, namely:

$$\mathfrak{h} := \operatorname{Lie} P^{\alpha_1} \oplus \mathfrak{g}_{-2\alpha_1-\alpha_2} \quad \text{and} \quad \mathfrak{l} := \operatorname{Lie} P^{\alpha_1} \oplus \mathfrak{g}_{-\alpha_1} \oplus \mathfrak{g}_{-\alpha_1-\alpha_2}.$$

Lemma 4.3.12. *The 2-Lie subalgebras of* $\operatorname{Lie} G$ *containing strictly* $\operatorname{Lie} P^{\alpha_1}$ *are exactly* $\mathfrak{h}$ *and* $\mathfrak{l}$.

Next, we consider the subgroups H and L of the group G, defined as being of height one with Lie algebra respectively equal to

$$\operatorname{Lie} H := \mathfrak{g}_{-2\alpha_1-\alpha_2} \quad \text{and} \quad \operatorname{Lie} L := \mathfrak{g}_{-\alpha_1} \oplus \mathfrak{g}_{-\alpha_1-\alpha_2}. \tag{4.3.20}$$

Finally, set

$$P_{\mathfrak{h}} := \langle H, P^{\alpha_1} \rangle \quad \text{and} \quad P_{\mathfrak{l}} := \langle L, P^{\alpha_1} \rangle.$$

This defines two parabolic subgroups which cannot be described as $(\ker \xi) P^{\alpha_1}$ for some isogeny ξ with source G, due to the fact that $\operatorname{Lie} G$ is simple (see [Str, 4.4]).

These two exotic subgroups are enough to complete the classification in type G_2. This is proven in Proposition 3.2.26; we recall here the precise statement for reference.

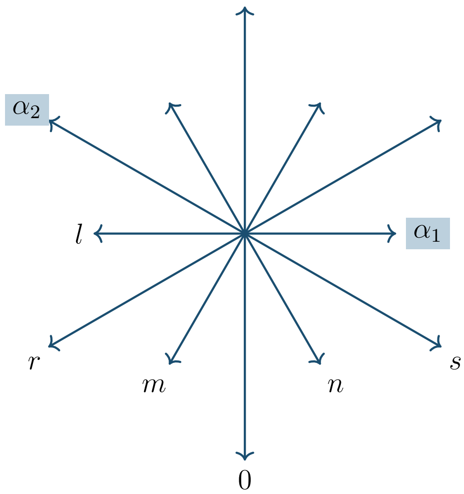
Proposition 4.3.13. *Let G be of type G_2 in characteristic two.*
Then all parabolic subgroups of G having P^{α_1} as reduced part are either of standard type, or obtained from $P_{\mathfrak{l}}$ and $P_{\mathfrak{h}}$ by pulling back with an iterated Frobenius homomorphism.

Proposition 4.3.14. *Let $P \subset G$ be a parabolic subgroup with*

$$P_{red} = B = P^{\alpha_1} \cap P^{\alpha_2}.$$

Then (4.1.1) is an equality.

Proof. Taking the quotient of P by the highest possible Frobenius kernel allows us to assume (4.3.19), as in the previous proof. The situation is summarized below: we place next to each negative root γ the height of the intersection $P \cap U_\gamma$, so that l, m, n, r, s are non-negative integers on which we now determine some conditions.



First, we notice that

$$[\mathfrak{g}_{-3\alpha_1-\alpha_2}, \mathfrak{g}_{-\alpha_2}] = [\mathfrak{g}_{-2\alpha_1-\alpha_2}, \mathfrak{g}_{-\alpha_1-\alpha_2}] = \mathfrak{g}_{-3\alpha_1-2\alpha_2}$$

because the two structure constants are respectively ± 1 and ± 3. Thus, one among r and s must vanish, and analogously for m and n.

• Assume that r vanishes and n does not, which yields that m is equal to zero. Moreover, l is nonzero because

$$[\mathfrak{g}_{-\alpha_1-\alpha_2}, \mathfrak{g}_{\alpha_2}] = \mathfrak{g}_{-\alpha_1}.$$

A direct computation, involving the root subgroups computed in Chapter 6, Remark 6.2.1, shows that for $a, b \in \mathbf{G}_a$,

$$(u_{-\alpha_1}(a), u_{-\alpha_1-\alpha_2}(b)) = u_{-3\alpha_1-\alpha_2}(a^2 b)\, u_{-3\alpha_1-2\alpha_2}(ab^2). \tag{4.3.21}$$

By [**Wen**, Proposition 8], if l is bigger than 1, then the unipotent infinitesimal part U_P^-, whose definition is recalled in (2.3.1), has nontrivial intersection with $U_{-3\alpha_1-\alpha_2}$. This yields that r is nonzero, which contradicts our assumption. On the other hand, if $n \geq 2$, then P has nontrivial intersection with $U_{-3\alpha_1-2\alpha_2}$, contradicting (4.3.19). This means that $l = n = 1$, thus for any s we get

$$P = P_{\mathfrak{l}} \cap {}_sGP^{\alpha_2}.$$

• Next, assume that both r and n vanish. If m is also zero, then

$$P = {}_lGP^{\alpha_1} \cap P^{\alpha_2} \quad \text{or} \quad P = P^{\alpha_1} \cap {}_sGP^{\alpha_2}$$

which are standard. Thus we can assume m to be nonzero. Another direct computation shows that for $b \in \mathbf{G}_a$,

$$(u_{\alpha_1}(1), u_{-2\alpha_1-\alpha_2}(b)) = u_{\alpha_2}(b^2)\, u_{-3\alpha_1-2\alpha_2}(b), \tag{4.3.22}$$

hence if $m \geq 2$, then P has again nontrivial intersection with $U_{-3\alpha_1-2\alpha_2}$. Moreover, l must vanish too, because

$$[\mathfrak{g}_{-\alpha_1}, \mathfrak{g}_{-2\alpha_1-\alpha_2}] = \mathfrak{g}_{-3\alpha_1-\alpha_2}$$

hence if l does not vanish then the same holds for r. This gives for any s,

$$P = P_{\mathfrak{h}} \cap {}_sGP^{\alpha_2}.$$

• We are left with the case where r is nonzero, which implies that s must vanish. From the equality

$$[\mathfrak{g}_{-\alpha_1-\alpha_2}, \mathfrak{g}_{\alpha_1}] = \mathfrak{g}_{-\alpha_2},$$

we deduce that n is also equal to zero. Finally, a direct computation shows

$$(u_{-3\alpha_1-\alpha_2}(a), u_{\alpha_1}(b)) = u_{-2\alpha_1-\alpha_2}(a)\, u_{-\alpha_1-\alpha_2}(b)\, u_{-\alpha_2}(b).$$

By [**Wen**, Proposition 8], if r is nonzero then we have that $u_{-\alpha_2}(b)$ belongs to P for $b \in \boldsymbol{\alpha}_2$; however this is a contradiction with the fact that s is equal to zero. □

CHAPTER 5

Geometry of rational projective homogeneous varieties

ABSTRACT. Using the Białynicki-Birula decomposition, we give a combinatorial description of classes of divisors and of curves on rational homogeneous projective varieties. Then we describe a family of examples of new such varieties of Picard rank two. We conclude with a few results on their geometry; among those, we show that there are only finitely many non-isomorphic homogeneous varieties of a prescribed dimension such that their anti-canonical bundle is globally generated.

5.1. Curves and divisors on flag varieties

We give here an explicit basis for 1-cycles and divisors modulo numerical equivalence on a flag variety $X = G/P$ of any Picard rank, with stabilizer P not necessarily reduced. We do so by describing the cells of an appropriate Białynicki-Birula decomposition of X in terms of the root system of G and of the root system of a Levi subgroup of the reduced part of P.

5.1.1. Białynicki-Birula decomposition of a G-simple projective variety. Flag varieties are smooth, projective and equipped with a G-action with a unique closed orbit, hence they form a particular class of simple G-projective varieties (for short, G-simple varieties), as in [Bri2]. Let us review here the main definitions and results concerning the Białynicki-Birula decomposition of such varieties, then specialize to flag varieties. The original work on the subject is [BB]; for a scheme-theoretic statement see [Mil, Theorem 13.47].

Let us consider a smooth G-simple variety X and fix a co-character $\lambda\colon \mathbf{G}_m \to T$ such that

$$B = \{g \in G\colon \lim_{t\to 0} \lambda(t) g \lambda(t^{-1}) \text{ exists in } G\},$$

which is equivalent to the condition that $\langle \gamma, \lambda \rangle > 0$ for all $\gamma \in \Phi^+$. This implies in particular that the set of fixed points under the $\mathbf{G}_m$-action induced by λ coincides with the set X^T of T-fixed points. Recall that the fixed-point scheme X^T is smooth, see for example [Mil, Theorem 13.1]. For any connected component $Y \subset X^T$ there are an associated *positive* and a *negative stratum*, defined as

$$X^+(Y) := \{x \in X\colon \lim_{t\to 0} \lambda(t) \cdot x \in Y\} \quad \text{and} \quad X^-(Y) := \{x \in X\colon \lim_{t\to 0} \lambda(t^{-1}) \cdot x \in Y\},$$

equipped with morphisms

$$p^+\colon X^+(Y) \to Y, \quad x \mapsto \lim_{t\to 0} \lambda(t) \cdot x,$$
$$p^-\colon X^-(Y) \to Y, \quad x \mapsto \lim_{t\to 0} \lambda(t^{-1}) \cdot x.$$

For more details on the definition of limits and strata, see for example [**May**].

THEOREM 5.1.1 (Białynicki-Birula decomposition). *Let X be a smooth G-simple projective variety. Then the following hold:*

- *The variety X is the disjoint union of the positive (resp. negative) strata as Y ranges over the connected components of X^T.*
- *The morphisms p^+ and p^- are affine bundles.*
- *The strata $X^+(Y)$ and $X^-(Y)$ intersect transversally along Y.*

Let us remark that the assumption on λ implies that positive strata are B- invariant, while negative strata are B^--invariant. In particular, the unique open positive stratum X^+ is equal to $X^+(x^+)$ where x^+ is the unique B^--fixed point, and analogously the unique open negative stratum X^- is equal to $X^-(x^-)$ where x^- is the unique B-fixed point. Let us recall here the main results from [**Bri2**] in the case where X is smooth.

THEOREM 5.1.2. *Let X be a smooth G-simple projective variety, x^- its B-fixed point, $X^- = X^-(x^-)$ the open negative cell and $D_1, \ldots, D_r$ the irreducible components of $X \backslash X^-$.*

(1) $D_1, \ldots, D_r$ are globally generated Cartier divisors, whose linear equivalence classes form a basis of $\mathrm{Pic}(X)$.

(2) Every ample (resp. nef) divisor on X is linearly equivalent to a unique linear combination of $D_1, \ldots, D_r$ with positive (resp. non-negative) integer coefficients. In particular, rational and numerical equivalence coincide on X i.e. the natural map $\mathrm{Pic}(X) \to N^1(X)$ is an isomorphism.

(3) There is a unique T-fixed point x_i^- such that D_i is the closure of $X^-(x_i^-)$. Moreover, x_i^- is isolated.

(4) Consider the B-invariant curve $C_i := \overline{B \cdot x_i^-}$. Then C_j intersects transversally D_j and no other; in particular, the intersection number satisfy

$$D_i \cdot C_j = \delta_{ij}.$$

(5) The convex cone of curves $\mathrm{NE}(X)$ is generated by the classes of $C_1, \ldots, C_r$, which form a basis of the rational vector space $N_1(X)_{\mathbf{Q}}$.

5.1.2. Białynicki-Birula decomposition of flag varieties. Let us now specialize to our case i.e. interpret the results of the above Section in terms of root systems. The first step consists in recalling the Bruhat decomposition of a flag variety with reduced stabilizer, i.e. $X = G/P_I$ where $I \subset \Delta$ is a basis for the root system of a Levi subgroup of P_I. In particular, for a simple root α the subgroup P^α - which has been widely used in the previous Sections - coincides with $P_{\Delta \backslash \{\alpha\}}$. Let us fix a set of representatives $\dot{w} \in N_G(T)$, for $w \in W = W(G,T)$ and let us recall the following (see [**Spr**, 8.3]).

THEOREM 5.1.3 (Bruhat decomposition). *Let $G \supset B \supset T$ be a reductive group, a Borel subgroup and a maximal torus, and $W = W(G,T)$. Then the following hold.*

(1) G is the disjoint union of the double cosets BwB, for $w \in W$.

(2) Let Φ_w be the set of positive roots γ such that $w^{-1}\gamma$ is negative. Then

$$U_w := \prod_{\gamma \in \Phi_w} U_\gamma$$

is a subgroup of the unipotent radical of B, with the product being taken in any order.

(3) *The map $U_w \times B \to BwB$ given by $(u, b) \mapsto u\dot{w}b$ is an isomorphism of varieties.*

This gives a decomposition of G/B into the disjoint union of the cells BwB/B, which are isomorphic to U_w i.e. to affine spaces of dimension equal to the length of w. Since we want to work with G/P_I instead of G/B, we shall not consider the whole Weyl group but its quotient by the subgroup W_I generated by the reflections corresponding to simple roots in I.

Lemma 5.1.4. *In any left coset of W_I in W there is a unique element w characterized by the fact that $wI \subset \Phi^+$ or by the fact that the element w is of minimal length in wW_I.*

Proof. See [**BT**, Proposition 3.9]. □

We denote the set of such representatives as W^I. In particular, denoting w_0 and $w_{0,I}$ the element of longest length of W and W_I respectively, then $w_0^I := w_0 w_{0,I}$ is the element of longest length in W^I.

Proposition 5.1.5 (Generalized Bruhat decomposition)**.** *For a fixed $I \subset \Delta$, the group G is the disjoint union of the double cosets BwP_I, where w ranges over the set W^I.*

In order to get a similar statement as (3) in Theorem 5.1.3, let us consider for any $w \in W^I$ the sets

$$\Phi_w^I := \{\gamma \in \Phi^+ : w^{-1}\gamma \notin \Phi^+ \text{ and } w^{-1}\gamma \notin \Phi_I\}, \tag{5.1.1}$$

$$\Phi_{w,I} := \Phi_w \backslash \Phi_w^I = \Phi_w \cap \Phi_I^+. \tag{5.1.2}$$

Lemma 5.1.6. *With the above notation, let us fix $w \in W^I$.*

(1) *The groups U_γ, with γ ranging over Φ_w^I (resp. $\Phi_{w,I}$) generate two subgroups of the unipotent radical of B,*

$$U_w^I = \prod_{\gamma \in \Phi_w^I} U_\gamma \quad \text{and} \quad U_{w,I} = \prod_{\gamma \in \Phi_{w,I}} U_\gamma,$$

with the product being taken in any order.

(2) *The product map $U_w^I \times P_I \to BwP_I$ given by $(u, h) \mapsto u\dot{w}h$ is an isomorphism of varieties.*

Proof. To prove (1), let us recall that for any pair of roots $\gamma, \delta \in \Phi$ there exist constants c_{ij} such that

$$(u_\gamma(x), u_\delta(y)) = \prod_{i,j>0,\, i\gamma+j\delta \in \Phi} u_{i\gamma+j\delta}(c_{ij}x^i y^j), \qquad \text{for all } x, y \in \mathbf{G}_a$$

(see [**Spr**, Proposition 8.2.3]). If γ and δ are both in Φ_w^I, then $w^{-1}(i\gamma+j\delta)$ is still negative and not belonging to Φ_I, hence by (5.1.1) the product of the root subgroups with roots ranging over Φ_w^I is a group. The same reasoning holds for the second product.
Moving on to (2), let us consider an element $x \in BwP_I$. Let us fix an order on $\Phi_w^I = \{\gamma_1, \dots, \gamma_l\}$ and on $\Phi_{w,I} = \{\delta_1, \dots, \delta_m\}$. By Theorem 5.1.3, there are a unique $w' \in W_I$, a unique $u = u_{\gamma_1}(x_1) \cdot \ldots \cdot u_{\gamma_l}(x_l) \in U_w^I$, a unique $u' = u_{\delta_1}(y_1) \cdot \ldots \cdot u_{\delta_m}(y_m) \in U_{w,I}$ and

a unique $b \in B$ such that $x = uu'\dot{w}\dot{w}'b \in Bww'B$. Moreover, by [Spr, 8.1.12(2)], there exist constants $c_i \in k$ such that

$$u'\dot{w} = \left(\prod_{i=1}^{m} u_{\delta_i}(y_i)\right)\dot{w} = \dot{w}\left(\prod_{i=1}^{m} \dot{w}^{-1}u_{\delta_i}(y_i)\dot{w}\right) = \dot{w}\prod_{i=1}^{m} u_{w^{-1}\delta_i}(c_i y_i) =: \dot{w}u''$$

Since $w^{-1}\delta_i$ is in Φ_I for all i, the product u'' is an element of P_I, as well as $h := u''\dot{w}'b$ because $w' \in W_I$. This gives a unique way to write x as product $u\dot{w}h$ for some $u \in U_w^I$ and $h \in P_I$. □

Next, let us go back to our original setting: consider a sequence $G \supset P \supset P_{\text{red}} = P_I \supset B \supset T$ and look at the map

$$\widetilde{X} := G/P_I \xrightarrow{\quad\sigma\quad} G/P =: X,$$

in order to relate the geometry of X to that of $\widetilde{X}$. The morphism σ is finite, purely inseparable and hence a homeomorphism between the underlying topological spaces. Let us denote as $\tilde{o} \in \widetilde{X}$ and $o \in X$ the respective base points.

The decomposition of Proposition 5.1.5 allows us to express the variety $\widetilde{X}$ as the disjoint union of the cells $BwP_I/P_I = Bw\tilde{o}$ as $w \in W^I$. Let us remark that W^I corresponds to the set of isolated points under the T-action, i.e. that

$$(\widetilde{X})^T = \{w\tilde{o} : w \in W/W_I\}$$

and the same holds for X. It is hence natural if such a decomposition coincides with the Białynicki-Birula decomposition of Theorem 5.1.1. This is useful because the advantage of the first one is that it is more explicit and easier to manipulate, while the second can be defined also on X, independently of the smoothness of the stabilizer. Let us denote as $\widetilde{X}_w^+$ (resp. X_w^+) the positive Białynicki-Birula strata associated to the T-fixed point $w\tilde{o}$ (resp. wo), and the analogous notation for negative strata.

Lemma 5.1.7. *For any $w \in W/W_I$, we have*

$$Bw\tilde{o} = \widetilde{X}_w^+ \quad \textit{and} \quad Bwo = X_w^+.$$

PROOF. For the first equality, $w\tilde{o}$ belongs to $\widetilde{X}_w^+$ because it is a T-fixed point. Moreover, positive strata are B-invariant which means that $Bw\tilde{o} \subseteq \widetilde{X}_w^+$. The other inclusion comes from the fact that $\widetilde{X}$ can be expressed as the disjoint union of both the strata of the two decompositions with the same index set.
Next, let us consider $Bwo = \sigma(Bw\tilde{o})$, which equals $\sigma(\widetilde{X}_w^+)$ by what we just proved. The inclusion $\sigma(\widetilde{X}_w^+) \subset X_w^+$ comes from the fact that σ being T-equivariant respects the Białynicki-Birula decomposition, while the other inclusion is due to the fact that

$$\bigsqcup_{w \in W^I} Bwo = X = \bigsqcup_{w \in W^I} X_w^+.$$

because σ is an homeomorphism. □

Remark 5.1.8. How can we visualize the morphism σ on cells? By Proposition 5.1.5, the Bruhat cell associated to $w \in W^I$ in $\widetilde{X}$ is an affine space of dimension l, equal to the cardinality of $\Phi^I_w = \{\gamma_1, \ldots, \gamma_l\}$. Let us consider the integers n_i, which we recall are associated to the roots in Φ^I_w via the equality

$$U_{-\gamma_i} \cap P = u_{-\gamma_i}(\alpha_{p^{n_i}}).$$

If we denote as Y_i the coordinate on the affine line given by U_{γ_i}, then the morphism σ acts on such a line as an n_i-th iterated Frobenius morphism, hence its behavior on the cell $Bw\tilde{o} = \widetilde{X}^+_w$ can be summarized in the following diagram

$$\begin{array}{ccc} U^I_w \simeq \widetilde{X}^+_w = \operatorname{Spec} k[Y_1, \ldots, Y_l] \simeq \mathbf{A}^l & \hookrightarrow & G/P_I \\ \downarrow \sigma & & \downarrow \sigma \\ X^+_w = \operatorname{Spec} k[Y_1^{p^{n_1}}, \ldots, Y_l^{p^{n_l}}] \simeq \mathbf{A}^l & \hookrightarrow & G/P \end{array}$$

We reinterpret all the ingredients of Theorem 5.1.2 in order to specialize and state it in the case of flag varieties. First, $X = G/P$ is indeed smooth, projective and G-simple. Its unique B-fixed point is $x^- = o$ the base point, which gives as open cell $B^-o = Bw_0o = Bw^I_0o = X^+_{w^I_0}$. Moreover, the irreducible components of $X \backslash X_{w^I_0}$ are the closures of the strata of codimension one, i.e. the cells Bwo with $w \in W^I$ of length $l(w) = l(w^I_0) - 1$. Those are exactly of the form $w = w_0 s_\alpha w_{0,I}$ for $\alpha \in \Delta \backslash I$, since for $\alpha \in I$ we have that $w_0 s_\alpha$ is in the same left coset as w^I_0. In particular, the divisors in the statement of Theorem 5.1.2 are

$$D_\alpha = \overline{Bw_0 s_\alpha w_{0,I} o} = \overline{Bw_0 s_\alpha o} = \overline{B^- s_\alpha o}, \quad \text{for } \alpha \in \Delta \backslash I,$$

hence the unique T-fixed point x^-_α such that D_α is the closure of $X^-(x^-_\alpha)$ is $x^-_\alpha = s_\alpha o$, and we are led to consider the B-invariant curves

$$C_\alpha = \overline{Bx^-_\alpha} = \overline{Bs_\alpha o}.$$

We are now able to reformulate the results of Section 5.1.1 in the following:

THEOREM 5.1.9. *Let us consider a sequence $G \supset P \supset P_{red} = P_I \supset B \supset T$ and let $X = G/P$ with base point o and open cell $X^- = B^-o$. Then the following hold:*

(1) *The irreducible components of $X \backslash X^-$ are the closures D_α of the negative cells associated to the points $s_\alpha o$ for $\alpha \in \Delta \backslash I$. Moreover, they are globally generated Cartier divisors, whose linear equivalence classes form a basis for* $\operatorname{Pic}(X)$.
(2) *Every ample (resp. nef) divisor on X is linearly equivalent to a unique linear combination of the D_α's with positive (resp. non-negative) integer coefficients. In particular, the natural map* $\operatorname{Pic}(X) \to N^1(X)$ *is an isomorphism.*
(3) *Considering the B-invariant curves C_α's defined above, the intersection numbers satisfy $D_\alpha \cdot C_\beta = \delta_{\alpha\beta}$.*
(4) *The convex cone of curves* $\operatorname{NE}(X)$ *is generated by the classes of the C_α's, which form a basis of the rational vector space $N_1(X)_{\mathbf{Q}}$.*

5.2. Contractions

Theorem 5.1.9 tells us in particular that the Picard group of a flag variety $X = G/P$ is a free **Z**-module of rank the number of simple roots not belonging to the root system of a Levi factor of P_{red}. This gives a motivation to the study, done in Chapter 3, of parabolic subgroups having maximal reduced part. In order to move on to higher ranks by exploiting the previous results in rank one, we adopt the following strategy : we define a finite collection of morphisms which behave nicely, arise naturally from the variety X, and whose targets are homogeneous spaces of Picard rank one. As a first step towards such a construction, we recall the notion of a contraction between varieties and some of its properties.

Definition 5.2.1. Let X and Y be varieties over an algebraically closed field k. A *contraction* between them is a proper morphism $f \colon X \to Y$ such that $f^\sharp \colon \mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X$ is an isomorphism.

We will make use of the following results, stated here for reference. They correspond respectively to [**Deb**, Proposition 1.14] and to [**Bri3**, 7.2].

THEOREM 5.2.2. *Let $f \colon X \to Y$ be a contraction between projective varieties over k. Then f is uniquely determined, up to isomorphism, by the convex subcone* $\mathrm{NE}(f)$ *of* $\mathrm{NE}(X)$ *generated by the classes of curves which it contracts. Moreover, if Y' is a third projective variety and $f' \colon X \to Y'$ satisfies* $\mathrm{NE}(f) \subset \mathrm{NE}(f')$, *then there is a unique morphism $\psi \colon Y \to Y'$ such that $f' = \psi \circ f$.*

$$\begin{array}{ccc} X & \xrightarrow{\;f\;} & Y \\ & {\scriptstyle f'}\searrow \quad \swarrow{\scriptstyle \psi} & \\ & Y' & \end{array}$$

THEOREM 5.2.3 (Blanchard's Lemma). *Let $f \colon X \to Y$ be a contraction. Assume that X is equipped with an action of a connected algebraic group G. Then there exists a unique G-action on Y such that the morphism f is G-equivariant.*

The following construction is done here for any globally generated line bundle and is then applied to $\mathcal{O}_X(D_\alpha)$ to define the desired family of contractions.

Lemma 5.2.4. *Let X be a projective variety over k and $\mathcal{L}$ a line bundle over X which is generated by its global sections. Then*

(a) There is a well-defined contraction

$$f \colon X \longrightarrow Y := \mathrm{Proj} \bigoplus_{n=0}^{\infty} H^0(X, \mathcal{L}^{\otimes n}).$$

(b) The sheaf $\mathcal{O}_Y(1)$ is invertible and ample, and $\mathcal{L} = f^*\mathcal{O}_Y(1)$.

(c) A curve C in X is contracted by f if and only if $\mathcal{L} \cdot C = 0$.

PROOF. (a) : Let us denote as S the graded ring on the right hand side and denote as $S_d = H^0(X, \mathcal{L}^{\otimes d})$ its homogeneous part of degree d. The scheme X is covered by the open subsets

$$X_t := \{x \in X,\, t_x \notin \mathfrak{m}_x\mathcal{L}_x\} = X \backslash Z(t),$$

where t ranges over the nonzero elements of S_1, because by hypothesis $\mathcal{L}$ is globally generated. On the other hand, the scheme $Y = \operatorname{Proj} S$ is covered by the open subset

$$D(t) = \operatorname{Spec}\left(\bigcup_{n=0}^{\infty} \frac{H^0(X, \mathcal{L}^{\otimes nd})}{t^n}\right),$$

where $t \in S_d$ for some $d > 0$, by definition of the Proj. This allows to define f via the inclusion

$$\bigcup_{n=0}^{\infty} \frac{H^0(X, \mathcal{L}^{\otimes nd})}{t^n} \subset \mathcal{O}_X(X_t), \quad \text{for } t \in S_d. \tag{5.2.1}$$

Moreover, [**Har**, II, Lemma 5.14], applied to the coherent sheaf $\mathcal{O}_X$ and the line bundle $\mathcal{L}^{\otimes nd}$, implies that (5.2.1) is an equality, which gives the condition $f_*\mathcal{O}_X \simeq \mathcal{O}_Y$.

(b) : Let us consider the sheaf $\mathcal{O}_Y(1)$ defined as in [**Har**, II, Proposition 5.11] and let us fix some global section $s \in H^0(X, \mathcal{L})$. Since we have the trivialisation $\mathcal{L}_{|X_s} \simeq s\mathcal{O}_{X_s}$, considering sections over X_s gives

$$H^0(X_s, \mathcal{L}) = \frac{s^{n+1}\mathcal{O}_X(X_s)}{s^n} = \bigcup_{n=0}^{\infty} \frac{H^0(X, \mathcal{L}^{\otimes(n+1)})}{s^n} = H^0(D(s), \mathcal{O}_Y(1)).$$

Next, let $V = H^0(X, \mathcal{L})^*$; since $\mathcal{L}$ is globally generated, we have a morphism $g\colon X \to \mathbf{P}(V)$ such that $\mathcal{L} = g^*\mathcal{O}_{\mathbf{P}(V)}(1)$. Consider the Stein factorisation of g as

$$X \xrightarrow{\ \varphi\ } Z \xrightarrow{\ \psi\ } \mathbf{P}(V)$$

where the morphism φ satisfies $\varphi_*\mathcal{O}_X = \mathcal{O}_Z$, while the map ψ is finite. Let $\mathcal{M} := \psi^*\mathcal{O}_{\mathbf{P}(V)}(1)$; then $\mathcal{M}$ is an ample invertible sheaf on Z, satisfying $\mathcal{L} = \varphi^*\mathcal{M}$. By the projection formula, one has

$$H^0(X, \mathcal{L}^{\otimes n}) = H^0(Z, \mathcal{M}^{\otimes n}) \quad \text{for all } n,$$

which implies that $Y = Z$ and that $\mathcal{L} = f^*\mathcal{M}$. In particular, Y is covered by the $D(s)$, where s ranges over the nonzero elements of $V^* = H^0(X, \mathcal{L}) = S_1$. By applying again the projection formula,

$$H^0(D(s), \mathcal{O}_Y(1)) = H^0(X_s, \mathcal{L}) = H^0(D(s), \mathcal{M}) \quad \text{for all } n,$$

so we get that $\mathcal{M} = \mathcal{O}_Y(1)$, as wanted.

(c) : We just proved that $\mathcal{O}_Y(1)$ is invertible and ample, thus it must have strictly positive intersection with any non-zero effective 1-cycle. In other words, given a nonzero class $C \in \operatorname{NE}(X)$, $f_*C = 0$ if and only if

$$0 = \mathcal{O}_Y(1) \cdot f_*C = f^*\mathcal{O}_Y(1) \cdot C = \mathcal{L} \cdot C,$$

by the projection formula, and we are done. □

Before going back to our particular case, let us prove a criterion for a morphism between homogeneous spaces to be a contraction.

Lemma 5.2.5. *Consider a chain of algebraic groups $H \subset H' \subset G$ over k. The morphism $f\colon G/H \longrightarrow G/H'$ is a contraction if and only if H'/H is proper over k and $\mathcal{O}(H'/H) = k$.*

Proof. Let us consider $q\colon G \to G/H$ and $q'\colon G \to G/H'$ to be the quotient maps and $m\colon G \times H/H' \to G/H$ the morphism given by the group multiplication and then by quotienting by H: by [Mil, Proposition 7.15] we have a cartesian square

$$\begin{array}{ccc} G \times H'/H & \xrightarrow{pr_G} & G \\ \downarrow m & & \downarrow q' \\ G/H & \xrightarrow{f} & G/H' \end{array}$$

Since q' is faithfully flat and pr_G is obtained as base change of f via such a morphism, f being proper is equivalent to pr_G being proper; now, the latter is obtained as base change of $H'/H \to \operatorname{Spec} k$ via the structural morphism of G, which is also fppf, hence it is proper if and only if H'/H is proper over k. This shows the first condition.
Moreover, the formation of the direct image of sheaves also commutes with fppf extensions: more precisely, applying this to the structural sheaves in our case yields

$$(q')^* f_* \mathcal{O}_{G/H} = (pr_G)_* \mathcal{O}_{G \times H'/H} = \mathcal{O}_G \otimes \mathcal{O}_{H'/H}(H'/H),$$

hence by taking $(q'_*)^H$ on both sides one gets

$$f_* \mathcal{O}_{G/H} = \mathcal{O}_{G/H'} \iff \mathcal{O}_{H'/H}(H'/H) = k,$$

which gives the second condition. □

Remark 5.2.6. Let us consider again a fixed parabolic subgroup P. We now construct a collection of morphisms $f_\alpha\colon X \to G/Q^\alpha$, for $\alpha \in \Delta\backslash I$, such that

(1) the target G/Q^α is defined in a concrete geometrical way,
(2) each f_α is a contraction,
(3) the stabilizer Q^α coincides with the smallest subgroup scheme of G containing both P and P^α: in particular, $(Q^\alpha)_{\text{red}}$ is a maximal reduced parabolic subgroup,
(4) the collection $(f_\alpha)_{\alpha\in\Delta\backslash I}$ "tells us a lot" about the variety X.

The reason why Q^α is not directly defined as being the algebraic subgroup generated by P and P^α is that this notion does not behave well since P is nonreduced in general. Let us apply Lemma 5.2.4 to the variety $X = G/P$ and the line bundle $\mathcal{L} = \mathcal{O}_X(D_\alpha)$, which can be done thanks to Theorem 5.1.9. This gives a contraction

$$f_\alpha\colon X \longrightarrow Y_\alpha := \operatorname{Proj} \bigoplus_{n=0}^{\infty} H^0(X, \mathcal{O}_X(nD_\alpha)). \tag{5.2.2}$$

By Theorem 5.2.3, there is a unique G-action on Y_α such that f_α is equivariant. Moreover, since f_α is a dominant morphism between projective varieties, it is surjective, hence the target must be of the form $Y_\alpha = G/Q^\alpha$ for some subgroup scheme $P \subseteq Q^\alpha \subsetneq G$. We take this construction as the definition of the subgroup Q^α, so that conditions (1) and (2) are already satisfied. Moreover, by Theorem 5.1.9 and Lemma 5.2.4, a curve C is contracted by f_α if and only if $D_\alpha \cdot C = 0$, meaning that this map contracts all C_β for $\beta \neq \alpha$ while it restricts to a finite morphism on C_α. This leaves one more condition to show.

Lemma 5.2.7. *The smallest subgroup scheme of G containing both P and P^α is Q^α.*

Proof. By definition of Y_α we have the inclusion $P \subset Q^\alpha$.
Let H be the subgroup scheme of G generated by P and P^α. Since

$$P_{\mathrm{red}} = P_I = \bigcap_{\alpha \in \Delta \setminus I} P^\alpha,$$

the subgroup generated by P_{red} and P^α is just P^α. Next, consider the quotient map $\widetilde{\pi}\colon \widetilde{X} \to G/P^\alpha$ and the composition $f_\alpha \circ \sigma\colon \widetilde{X} \to G/Q^\alpha$: the latter contracts, by the above discussions, all curves $\widetilde{C}_\beta$ for $\beta \neq \alpha$, hence $\mathrm{NE}(\widetilde{\pi}) \subset \mathrm{NE}(f_\alpha \circ \sigma)$. Moreover, $\widetilde{\pi}$ is a contraction by Lemma 5.2.5, because its fiber at the base point is P^α/P_I which is proper and has no nonconstant global regular functions. By Theorem 5.2.2, there exists a unique morphism φ making the diagram

$$\begin{array}{ccc} \widetilde{X} = G/P_{\mathrm{red}} & \xrightarrow{\ \widetilde{\pi}\ } & G/P^\alpha \\ \downarrow{\scriptstyle \sigma} & & \vdots{\scriptstyle \varphi} \\ X = G/P & \xrightarrow{\ f_\alpha\ } & G/Q^\alpha \end{array}$$

commute: this shows $P^\alpha \subset Q^\alpha$ hence $H \subset Q^\alpha$.
Conversely, let us consider the projection $\pi\colon X \to G/H$. We already know by Theorem 5.1.9 that $\widetilde{\pi}$ contracts all $\widetilde{C}_\beta$ for $\beta \neq \alpha$; moreover, the square on the left in the following diagram is commutative and its horizontal arrows are both homeomorphisms. This implies that π contracts all C_β for $\beta \neq \alpha$. In other words, the inclusion $\mathrm{NE}(f_\alpha) \subset \mathrm{NE}(\pi)$ holds.

$$\begin{array}{ccccc} \widetilde{X} & \xrightarrow{\ \sigma\ } & X & \xrightarrow{\ f_\alpha\ } & G/Q^\alpha \\ \downarrow{\scriptstyle \widetilde{\pi}} & & \downarrow{\scriptstyle \pi} & \swarrow{\scriptstyle \psi} & \\ G/P^\alpha & \longrightarrow & G/H & & \end{array}$$

Since f_α is a contraction by definition, this gives a factorisation by ψ - again by Theorem 5.2.2 - which means that $Q^\alpha \subset H$. ☐

Remark 5.2.8. The homogeneous space X is now equipped with a finite number of contractions f_α such that the target of each morphism has Picard group $\mathbf{Z}$, with a unique canonical ample generator, corresponding to the image of D_α. The inclusion

$$P \subseteq \bigcap_{\alpha \in \Delta} Q^\alpha \tag{5.2.3}$$

holds by definition of Q^α. If the characteristic is $p \geq 5$, by [**Wen**] there are nonnegative integers m_α for $\alpha \in \Delta \setminus I$ such that P is the intersection of the ${}_{m_\alpha}GP^\alpha$, hence

$$P \subset Q^\alpha \subset {}_{m_\alpha}GP^\alpha$$

and the inclusion (5.2.3) becomes an equality. Geometrically, this corresponds to saying that the product map

$$f := \prod_{\alpha \in \Delta} f_\alpha \colon X \longrightarrow \prod_{\alpha \in \Delta} G/Q^\alpha$$

is a closed immersion, realizing X as the unique closed orbit of the G-action on the target.

5.3. Examples in Picard rank two

Let us consider a simple simply connected algebraic group G over k, having Dynkin diagram with an edge of multiplicity equal to the characteristic $p \in \{2,3\}$, so that the definitions and properties of Section 2.5.1.1 apply. In particular, under such assumption we have the notion of very special isogeny for a simple, simply connected group. Let us recall that a parabolic subgroup is said to be *of standard type* if it is of the form ${}_{m_1}GP^{\alpha_1} \cap \ldots \cap {}_{m_r}GP^{\alpha_r}$ for some integers m_i and simple roots α_i. Analogously a homogeneous variety is said to be *of standard type* if its underlying variety is isomorphic to some G'/P', where P' is a parabolic subgroup of standard type.

The main result in this Section is the following, which provides us with a first family of homogeneous projective varieties (in types B_n, C_n and F_4) which are not of standard type.

Proposition 5.3.1. *Let $p = 2$ and consider a simple, simply connected group G and two distinct simple roots α and β such that: either G is of type B_n or C_n and the pair (α, β) is of the form (α_j, α_i) with $i < j < n$ or $j = n$ and $i < n-1$, or G is of type F_4 and the pair (α, β) is one among*

$$(\alpha_1, \alpha_4), \quad (\alpha_2, \alpha_1), \quad (\alpha_2, \alpha_4), \quad (\alpha_3, \alpha_1), \quad (\alpha_3, \alpha_4), \quad (\alpha_4, \alpha_1).$$

Then the homogeneous space $X = G/({}_rN_GP^\alpha \cap P^\beta)$ is not *of standard type.*

First, we give a motivation to the fact that we look for an example in rank two, then we prove Proposition 5.3.1 in two consecutive steps.

Let us fix a simple root $\alpha \in \Delta$. In order to find a parabolic subgroup not of standard type, the easiest and more natural idea is to consider the very special isogeny $\pi_G \colon G \longrightarrow \overline{G}$ and the subgroup $P := N_GP^\alpha$. Its reduced part $P_{\mathrm{red}} = P^\alpha$ is maximal, but P is not of the form ${}_mGP^\alpha$ for any m. Indeed, its associated function $\varphi_P \colon \Phi^+ \to \mathbf{N} \cup \{\infty\}$ is given by

$$\begin{aligned} \gamma &\longmapsto \infty && \text{if } \alpha \notin \mathrm{Supp}(\gamma) \\ \gamma &\longmapsto 0 && \text{if } \alpha \in \mathrm{Supp}(\gamma) \text{ and } \gamma \in \Phi_> \\ \gamma &\longmapsto 1 && \text{if } \alpha \in \mathrm{Supp}(\gamma) \text{ and } \gamma \in \Phi_< \end{aligned}$$

while the function associated to a parabolic subgroup of standard type satisfies $\varphi_{{}_mGP^\alpha}(\gamma) = m$ for all roots γ containing α in their support, regardless of their length. There always exist both a short and a long root containing any simple root α in their support, namely

(5.3.1) • in type B_n, $\mathrm{Supp}(\varepsilon_1) = \mathrm{Supp}(\varepsilon_1 + \varepsilon_2) = \Delta$;

(5.3.2) • in type C_n, $\mathrm{Supp}(2\varepsilon_1) = \mathrm{Supp}(\varepsilon_1 + \varepsilon_2) = \Delta$;

(5.3.3) • in type F_4, $\mathrm{Supp}(\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4) = \mathrm{Supp}(\alpha_1 + 2\alpha_2 + 4\alpha_3 + 2\alpha_4) = \Delta$;

(5.3.4) • in type G_2, $\mathrm{Supp}(2\alpha_1 + \alpha_2) = \mathrm{Supp}(3\alpha_1 + 2\alpha_2) = \Delta$.

Let us remark that the above roots can be constructed in a uniform way: they are respectively the highest short root and the highest (long) root. Thus, we can conclude that $\varphi_P \neq \varphi_{mGQ}$ for all m, proving that P is a parabolic subgroup not of standard type. However, $X = G/P$ is isomorphic as a variety to $\overline{G}/P_{\overline{\alpha}}$, hence the homogeneous space X is still of standard type.

The same reasoning applies when one considers the product of a parabolic subgroup of standard type and of a kernel of a noncentral isogeny with source G: this might define a new parabolic subgroup, but an homogeneous space which is still of standard type. Together with Theorem 3.3.2, this implies that it is not possible to find examples of homogeneous spaces not of standard type having Picard rank one, when the characteristic satisfies the edge hypothesis. This provides a motivation to the study of the rank two case, which means considering parabolic subgroups whose reduced part is of the form

$$P^\alpha \cap P^\beta$$

for two distinct simple roots α and β. In such a context we are able to find the desired class of examples.

Lemma 5.3.2. *Let us consider a simple, simply connected group G having Dynkin diagram with an edge of multiplicity p, fix two distinct simple roots α and β and an integer $r \geq 0$. Both the parabolic*

$$P := {}_rN_G P^\alpha \cap P^\beta$$

and its pull-back via the very special isogeny $\pi_{\overline{G}} \colon \overline{G} \to G$ are not *of standard type if and only if one of the following conditions is satisfied :*

- *(i) G is of type B_n or C_n and the pair (α, β) is of the form (α_j, α_i) with $i < j < n$ or $j = n$ and $i < n-1$;*
- *(ii) G is of type F_4 and the pair (α, β) is one amongst*

$$(\alpha_1, \alpha_4), \quad (\alpha_2, \alpha_1), \quad (\alpha_2, \alpha_4), \quad (\alpha_3, \alpha_1), \quad (\alpha_3, \alpha_4), \quad (\alpha_4, \alpha_1).$$

In particular, this situation can only happen when $p = 2$.

Proof. Let us take a look at the function

$$\varphi_P \colon \Phi^+ \to \mathbf{N} \cup \{\infty\}$$

associated to the parabolic P. Recall that it is determined by the equality

$$U_{-\gamma} \cap P = u_{-\gamma}(\alpha_{p^{\varphi(\gamma)}}), \quad \gamma \in \Phi^+.$$

Let us compare the numerical function φ_P to the analogous numerical function φ_Q, associated to some $Q = {}_mGP^\alpha \cap {}_nGP^\beta$ (i.e. a parabolic of standard type), which is necessarily of this form because $Q_{\text{red}} = P_{\text{red}} = P^\alpha \cap P^\beta$. Our aim is to find in which cases there is a contradiction with the equality $P = Q$. First of all, assuming $\varphi_P(\beta) = \varphi_Q(\beta)$ leads to $n = 0$. Now, let us write down the values that φ_P and φ_Q assume on all positive roots in the following table.

	$\alpha, \beta \in \text{Supp}(\gamma)$	$\alpha \in \text{Supp}(\gamma)$, $\beta \notin \text{Supp}(\gamma)$, γ short	$\alpha \in \text{Supp}(\gamma)$, $\beta \notin \text{Supp}(\gamma)$, γ long	$\beta \in \text{Supp}(\gamma)$
$\varphi_Q(\gamma)$	∞	m	m	0
$\varphi_P(\gamma)$	∞	$r+1$	r	0

Thus, the two functions are distinct if and only if there exist at least one long root and one short root containing α and not β in their respective supports. Let us examine each root system to determine when this is the case.

- If G is of type G_2 in characteristic $p = 3$, then all roots distinct from α_1 and α_2 contain both simple roots in their support, hence the desired condition is never satisfied. Thus from now on we can assume that $p = 2$.
- If G is of type B_n, let $\alpha = \alpha_j$ and $\beta = \alpha_i$ for some $1 \leq i, j \leq n$. A positive short root is of the form $\varepsilon_m = \alpha_m + \ldots + \alpha_{n-1} + 2\alpha_n$ for $m < n$ or $\varepsilon_n = \alpha_n$: hence if $j < i$ then a short root containing α in its support also contains β. Let us then assume $i < j$: in this case $\gamma = \varepsilon_j$ satisfies the condition. Moving on to long roots, if $i < j < n$ then $\gamma = \alpha = \varepsilon_j - \varepsilon_{j+1}$ is as wanted, while if $j = n$ then $\gamma = \varepsilon_{n-1} + \varepsilon_n = \alpha_n + 2\alpha_{n-1}$ satisfies the condition when $i < n-1$, while if $i = n-1$ then there is no such γ.
- If G is of type C_n, let $\alpha = \alpha_j$ and $\beta = \alpha_i$ for some $1 \leq i, j \leq n$. A positive long root is of the form $2\varepsilon_m = 2(\alpha_m + \ldots + \alpha_{n-1} + \alpha_n)$ for $m < n$ or $2\varepsilon_n = \alpha_n$: hence if $j < i$ then a long root containing α in its support also contains β. Let us then assume $i < j$: in this case $\gamma = 2\varepsilon_j$ satisfies the condition. Moving on to short roots, if $i < j < n$ then $\gamma = \alpha = \varepsilon_j - \varepsilon_{j+1}$ is as wanted, while if $j = n$ then $\gamma = \varepsilon_{n-1} + \varepsilon_n = \alpha_n + \alpha_{n-1}$ satisfies the condition when $i < n-1$, while if $i = n-1$ then there is no such γ. This completes condition (i).
- If G is of type F_4, there is no short root containing α_1 (resp. α_1, resp. α_2) in its support and not containing α_2 (resp. α_3, resp. α_3); moreover, there is no long root containing α_3 (resp. α_4, resp. α_4) in its support and not containing α_2 (resp. α_2, resp. α_3). This can be seen by directly looking at the list of positive roots in such a system, recalled at the beginning of Section 3.1.5. The remaining pairs are listed below, which gives condition (ii).

α	β	a short γ: $\alpha \in \mathrm{Supp}(\gamma), \beta \notin \mathrm{Supp}(\gamma)$	a long γ: $\alpha \in \mathrm{Supp}(\gamma), \beta \notin \mathrm{Supp}(\gamma)$
α_1	α_4	$\alpha_1 + \alpha_2 + \alpha_3$	α_1
α_2	α_1	$\alpha_2 + \alpha_3$	α_2
α_2	α_4	$\alpha_2 + \alpha_3$	α_2
α_3	α_1	α_3	$\alpha_2 + 2\alpha_3$
α_3	α_4	α_3	$\alpha_2 + 2\alpha_3$
α_4	α_1	α_4	$\alpha_2 + 2\alpha_3 + 2\alpha_4$

Up to this point we have only shown that the parabolic P is not of standard type if and only if conditions (i) or (ii) are satisfied. Now, let us consider the pull-back

$$\pi_{\overline{G}}^{-1}(P) = \pi_{\overline{G}}^{-1}({}_r N_G P^{\alpha} \cap P^{\beta}) = {}_{r+1}\overline{G} P^{\overline{\alpha}} \cap N_{\overline{G}} P^{\overline{\beta}}$$

and compare it with

$$Q = {}_m\overline{G} P^{\overline{\alpha}} \cap {}_n\overline{G} P^{\overline{\beta}},$$

analogously as before. This gives in particular, considering a root $\gamma \in \Phi^+$ satisfying $\overline{\alpha}, \overline{\beta} \in \mathrm{Supp}(\gamma)$, such that $\varphi_Q(\gamma) = \min(m, n)$ for all γ, while $\varphi_{\pi^{-1}(P)}(\gamma)$ is equal to 1 if γ is short, and equal to 0 if γ is long. To show that these two parabolics cannot coincide, it is enough to have both such a long and a short root. This is always the case, as recalled at the beginning of this Subsection in (5.3.1)- (5.3.3), hence this concludes the proof. □

Lemma 5.3.3. *Keeping the above notations, consider two distinct simple positive roots α and β satisfying one of the conditions of Lemma 5.3.2. Then the parabolic*

$$P := {}_rN_GP^\alpha \cap P^\beta$$

gives a variety $X := G/P$ which is not *of standard type.*

Proof. The reduced part of the parabolic subgroup P is $P_{\text{red}} = P^\alpha \cap P^\beta$: by Theorem 5.1.9, the convex cone of curves of the variety X is generated by the classes of the curves

$$C_\alpha = \overline{Bs_\alpha o} \quad \text{and} \quad C_\beta = \overline{Bs_\beta o}.$$

Next, let us consider the two contractions

$$f_\alpha \colon X \longrightarrow G/Q^\alpha \quad \text{and} \quad f_\beta \colon X \longrightarrow G/Q^\beta$$

defined by (5.2.2). Clearly, $Q^\beta = \langle P, P^\beta \rangle = P^\beta$ is smooth because $P \subset P^\beta$. On the other hand, let us show that $Q^\alpha = {}_rN_GP^\alpha$. Since both P and P^α are subgroups of the right hand term, the inclusion $Q^\alpha \subset {}_rN_GP^\alpha$ holds. To prove the other inclusion, let us notice that the hypothesis on α and β, as shown in the proof of Lemma 5.3.2, guarantees the existence of some short positive root γ containing α and not β in its support. In particular, this implies that

$$P \cap U_{-\gamma} = ({}_rN_GP^\alpha \cap U_{-\gamma}) \cap (P^\beta \cap U_{-\gamma}) = u_{-\gamma}(\alpha_{p^{r+1}}),$$

hence $Q^\alpha \cap U_{-\gamma}$ is the image of a Frobenius kernel of height at least equal to $r+1$. By the factorisation of isogenies in Proposition 2.5.12, the only two possibilities are thus $Q^\alpha = {}_{r+1}GP^\alpha$ and $Q^\alpha = {}_rN_GP^\alpha$, which allows to conclude that $Q^\alpha = {}_rN_GP^\alpha$. This means that the product of the contractions

$$f = f_\alpha \times f_\beta \colon X \hookrightarrow X_\alpha \times X_\beta \tag{5.3.5}$$

is a closed immersion, where X_α (resp. X_β) is the underlying variety of G/N_GP^α (resp. G/P^β). Moreover, these maps are - up to a permutation - uniquely determined by the variety X, because the monoid $\mathbf{N}C_\alpha \oplus \mathbf{N}C_\beta \subset N_1(X)$ of effective 1-cycles does not depend on the group action on it: the two contractions are uniquely determined by its two generators and by the fact that the first is a nonsmooth morphism while the second is smooth.
The following step consists in studying the automorphisms of the varieties X and X_β. First, let us consider the group $H := \underline{\mathrm{Aut}}^0_X$, which is a semisimple adjoint group. Its natural action on X gives, applying Theorem 5.2.3 to the contractions f_α and f_β respectively, two morphisms

$$\rho_\alpha \colon H \longrightarrow \underline{\mathrm{Aut}}^0_{X_\alpha} \quad \text{and} \quad \rho_\beta \colon H \longrightarrow \underline{\mathrm{Aut}}^0_{X_\beta},$$

which fit into the following commutative diagram.

$$\begin{array}{ccccc} G_{\mathrm{ad}} & \xrightarrow{\pi\times\mathrm{id}} & \overline{G}_{\mathrm{ad}} \times G_{\mathrm{ad}} & \hookrightarrow & \underline{\mathrm{Aut}}^0_{X_\alpha} \times \underline{\mathrm{Aut}}^0_{X_\beta} \\ & \searrow & & \nearrow_{\rho_\alpha\times\rho_\beta} & \\ & & H & & \end{array}$$

Let $Q \subset H$ be a parabolic subgroup satisfying $X = H/Q$. Since the variety X has Picard rank two, by Theorem 5.1.9 the group H is either simple or a product of two distinct simple factors $H_1 \times H_2$. Let us assume we are in the second case; then the reduced part of Q is determined by one simple root of H_1 and one simple root of H_2. By Lemma 4.1.6, there exist two parabolic subgroups $Q_1 \subset H_1$ and $Q_2 \subset H_2$ such that $(Q_1)_{\mathrm{red}} \times (Q_2)_{\mathrm{red}} = Q_{\mathrm{red}}$, and such that

$$X_\alpha = H_1/Q_1 \quad \text{and} \quad X_\beta = H_2/Q_2.$$

But then the group H would act transitively on the product $X_\alpha \times X_\beta$, which gives a contradiction with the embedding (5.3.5). Thus, the group H must be simple. Next, let us consider the automorphism group of X_β: except for a group of type C_n when $\beta = \alpha_1$, we can apply Theorem 3.2.3 to the variety $X_\beta = G/P^\beta$ since its stabilizer is smooth and since by Lemma 5.3.2 the pair $(G_{\mathrm{ad}}, P^\beta/Z(G))$ is not associated to any of the exceptional pairs. Thus we have

$$\underline{\mathrm{Aut}}^0_{X_\beta} = G_{\mathrm{ad}}.$$

In particular, ρ_β is a section of the inclusion of G_{ad} into $\underline{\mathrm{Aut}}^0_X$. This implies $H = K \times G_{\mathrm{ad}}$ for some subgroup $K \subset \underline{\mathrm{Aut}}^0_{X_\alpha}$. But then, K is contained in the centralizer $C_H(G_{\mathrm{ad}})$. The variety X being homogeneous under G_{ad}, we have that K fixes the base point o of X (because o is the only fixed point under the Borel subgroup $B \subset G$), thus it must fix all of X and hence it must be trivial. This means that $\underline{\mathrm{Aut}}^0_X = G_{\mathrm{ad}}$. We still have to treat the case $G = \mathrm{Sp}_{2n}$ and $\beta = \alpha_1$, for which Theorem 3.2.3 yields $\underline{\mathrm{Aut}}^0_{X_\beta} = \mathrm{PGL}_{2n}$. When considering X_α, we have

$$\underline{\mathrm{Aut}}^0_{X_\alpha} = \mathrm{SO}_{2n+1} \quad \text{if } \alpha = \alpha_j \quad \text{for } j < n,$$

while if $\alpha = \alpha_n$, then

$$\underline{\mathrm{Aut}}^0_{X_\alpha} = \underline{\mathrm{Aut}}_{\mathrm{SO}_{2n+1}/P^{\alpha_n}} = \mathrm{SO}_{2n+2},$$

again by Theorem 3.2.3. In both cases, the automorphism group of X_α embeds into SO_{2n+2}, and we get a commutative diagram as below.

$$\begin{array}{ccccc} G_{\mathrm{ad}} = \mathrm{PSp}_{2n} & \xrightarrow{\pi\times\mathrm{id}} & \mathrm{SO}_{2n+1} \times \mathrm{PSp}_{2n} & \hookrightarrow & \mathrm{SO}_{2n+2} \times \mathrm{PGL}_{2n} \\ & \searrow & & \nearrow & \\ & & H := \underline{\mathrm{Aut}}^0_X & & \end{array}$$

This yields that

$$G_{\mathrm{ad}} \subset H \subset \mathrm{PGL}_{2n} \tag{5.3.6}$$

and that the group H, up to an isogeny, is also contained in SO_{2n+1}. By pulling back (5.3.6) to the simply connected cover, we have that Sp_{2n} is contained in the connected component of the identity of H', where $H' \subset \mathrm{SL}_{2n}$ is the preimage of H. By [**MT**, Table 18.2], the subgroup Sp_{2n} is maximal among smooth connected subgroups of SL_{2n}; hence $(H')^0$ is either equal to Sp_{2n} or to SL_{2n}. This still allows for $H = \mathrm{PGL}_{2n}$, but the latter

for dimension reasons cannot embed up to an isogeny into SO_{2n+2}. Thus, we get that $H = \mathrm{PSp}_{2n} = G_{\mathrm{ad}}$ also in this case.
Finally, let us consider another action of a semisimple, simply connected G' onto the variety X; realizing it as a quotient G'/P' for some parabolic subgroup P'. Since it is simply connected, G' is either simple or the direct product $G_{(1)} \times \cdots \times G_{(l)}$ where each $G_{(i)}$ is simple.
• If G' is simple, then its action on X induces a morphism $G' \longrightarrow \underline{\mathrm{Aut}}^0_X = G_{\mathrm{ad}}$, which is in particular an isogeny. By Proposition 2.5.12, this morphism can be factorised as

$$G' \xrightarrow{F^m} G \twoheadrightarrow \underline{\mathrm{Aut}}^0_X \qquad \text{or} \qquad G' \xrightarrow{F^m \circ \pi} G \twoheadrightarrow \underline{\mathrm{Aut}}^0_X,$$

where the second possibility only can happen whenever G satisfies the edge hypothesis. The stabilizer of the G'-action is the preimage of the stabilizer of the G-action via such an isogeny, hence it is either of the form ${}_mGP$ for some m or of the form ${}_m\overline{G}\pi^{-1}(P)$. Now, a parabolic Q is of standard type if and only if ${}_mGQ$ is for any integer m, since the associated functions satisfy $\varphi_Q(\gamma) + m = \varphi_{{}_mGQ}(\gamma)$. This means that P' is of standard type if and only if P (resp. $\pi^{-1}(P)$) is. This remark, together with Lemma 5.3.2 allows us to conclude that, due to our choice of roots α and β, P' is still a parabolic subgroup not of standard type.
• If $G' = G_{(1)} \times \cdots \times G_{(l)}$ is not simple, consider the morphism

$$G_{(1)} \times \cdots \times G_{(l)} \xrightarrow{\phi} G \twoheadrightarrow G_{\mathrm{ad}}$$

determined by the action: then $H := \ker\phi$ is a normal subgroup of G' and the image of ϕ is simple, thus H is necessarily of the form

$$H = \prod_{i \neq i_0} G_{(i)} \times K, \quad \text{for some } K \subset Z(G_{(i_0)}),$$

thus K is trivial because the quotient G is also simply connected. In particular, denoting as $P_{(i_0)} := P' \cap G_{(i_0)}$, we have

$$X = G'/P' = G'/\left(\prod_{i \neq i_0} G_{(i)} \times P_{(i_0)}\right) = G_{(i_0)}/P_{(i_0)}$$

Applying the reasoning above to $G_{(i_0)}$ instead of G' leads to the conclusion that the associated function of $P_{(i_0)}$ is not of standard type, hence the same is true for the stabilizer $P' = \prod_{i \neq i_0} G_{(i)} \times P_{(i_0)}$. □

Notice that, except for the group of type G_2 in characteristic 2, Lemma 5.3.3 covers the classification of all homogeneous spaces of Picard rank two; thanks to Theorem 4.1.1. Indeed, Proposition 2.5.12 implies that one of the two kernels must be contained in the other, hence up to permuting α and β the inclusion $\ker\psi \subset \ker\varphi$ holds. Taking the quotient by $\ker\psi$ allows to assume either $P = {}_rGP^\alpha \cap P^\beta$, which is the standard type case, or $P = {}_rN_GP^\alpha \cap P^\beta$ for some $r \geq 0$. The latter gives a variety not of standard type if and only if $p = 2$ and the above hypothesis on roots is satisfied.

5.4. Geometric consequences in all ranks

Let us start by mentioning one immediate consequence of the complete classification of parabolic subgroups, stated in Theorem 4.1.1. This main result, together with the case of maximal reduced part (see Theorem 3.3.2) implies the following.

Corollary 5.4.1. *Except for a group of type G_2 in characteristic 2, any parabolic subgroup is of quasi-standard type.*

For the G_2 case, two *exotic* families of parabolic subgroups do occur: see Proposition 3.2.26 for more details.

The complete classification of non-reduced parabolic subgroups now allows us to start tackling a wide number of questions on the geometric properties of the corresponding homogeneous spaces. For instance, let us mention the class of SL_3-homogeneous hypersurfaces X_m in $\mathbf{P}^2 \times \mathbf{P}^2$, given by the equations

$$x_0 y_0^{p^m} + x_1 y_1^{p^m} + x_2 y_2^{p^m} = 0,$$

already cited in Example 2.3.14; these show that the varieties G/P are in general not Fano, since X_m is not Fano if $p^m > 3$. In [**Lau1**, Proposition 3.1], the canonical bundle is computed without assumption on the group nor the characteristic.

Let us move on to investigating some geometric properties of rational projective homogeneous spaces. In order to do it, we make use of the family of contractions of maximal relative Picard rank with source $X = G/P$, denoted

$$f_\alpha \colon X \longrightarrow G/Q^\alpha = G/\langle P, P^\alpha \rangle, \quad \text{for } \alpha \in \Delta \backslash I,$$

which are defined in (4.1.3). Those morphisms are entirely determined, up to a permutation, by the variety X: see Section 5.1 for more details.

Remark 5.4.2. Considering the underlying varieties, Theorem 4.1.1 implies that the product of the contractions f_α realises any rational homogeneous space X as a closed subvariety of the product

$$X = G/P \hookrightarrow \prod_{\alpha \in \Delta \backslash I} G/Q^\alpha \tag{5.4.1}$$

By Theorem 3.1.1, each G/Q^α is either isomorphic to a flag variety having as stabilizer a maximal reduced parabolic (hence defined over $\mathbf{Z}$) or, in the case of G_2 and characteristic 2, it can be isomorphic to the variety

$$\mathcal{X} := G_2/P_{\mathfrak{l}}.$$

The latter is described in Proposition 3.2.18 as being a general hyperplane section of the Sp_6-homogeneous variety of isotropic 3-dimensional subspaces in a 6-dimensional vector space.

Corollary 5.4.3. *Every ample line bundle on $X = G/P$ is very ample.*

Before proving this statement, let us briefly recall what ampleness on X looks like: thanks to the Białynicki-Birula decomposition, there is an explicit basis of the Picard group of X, given by the Schubert divisors D_α defined in (4.1.2). In Section 5.1, we show that a line bundle on X is ample if and only if it writes as a linear combination of the D_αs with strictly positive coefficients. In particular, from Corollary 5.4.3 we can deduce that X has a minimal embedding into projective space as

$$X \hookrightarrow \mathbf{P}(H^0(X, D)^\vee), \quad D := \sum_{\alpha \in \Delta \setminus I} D_\alpha.$$

Proof. (of Corollary 5.4.3) When the stabiliser P is reduced this is a well-known fact; it is also true for the exotic variety $\mathcal{X}$ above, due to its construction as a general hyperplane section of a generalised Sp_6-flag variety, with respect to the (very) ample generator of the Picard group. Thus, it holds for any rational projective homogeneous variety of Picard rank one. The embedding (5.4.1) into the product of the G/Q^α is enough to conclude the analogous statement for any X. □

5.4.1. Stabiliser of contractions: standard type. Let us now compute the stabilizer Q^α of each contraction f_α of maximal relative Picard rank, in the case of a parabolic subgroup P of standard type. This shows that parabolics of quasi-standard type - or even exotic ones in type G_2 - can already occur in this context.

Let P be of standard type and r be the Picard rank of G/P; then there are distinct simple roots $\beta_1, \ldots, \beta_r$ and non-negative integers $m_1 \leq \ldots \leq m_r$ satisfying

$$P = {}_{m_1}GP^{\beta_1} \cap \ldots \cap {}_{m_r}GP^{\beta_r}. \tag{5.4.2}$$

Considering the quotient of G by the m_1-th iterated Frobenius kernel allows us to assume that m_1 is equal to one, so that the contraction f_{β_1} associated to β_1 is smooth. Let us denote as

$$Q^i := Q^{\beta_i} = \langle P, P^{\beta_i} \rangle$$

the stabilizer of the target of the contraction associated to β_i; in particular,

$$Q^1 = P^{\beta_1}.$$

Clearly, by definition Q^i is contained in ${}_{m_i}GP^{\beta_i}$, however it could a priori be smaller; by Lemma 4.2.1, the intersection of Q^i with $U_{-\beta_i}$ necessarily has height m_i. We examine the different situations that can happen, in order to determine under which conditions Q^i is not of standard type anymore. First, let us introduce the following notation in type G_2.

Definition 5.4.4. Let G be of type G_2 in characteristic two.
The pullback by an m-th iterated Frobenius morphisms of the height one subgroup L, whose definition is recalled in (4.3.20), is denoted

$${}_mL := (F^m)^{-1}L.$$

Lemma 5.4.5. *Let P be a parabolic subgroup of standard type as in (5.4.2), with m_1 equal to one.*

(1) If G does not satisfy the edge hypothesis and is not of type G_2, then each Q^i is standard.

(2) If G satisfies the edge hypothesis,

$$Q^i = {}_{m_i-1}NP^{\beta_i}$$

if and only if all positive long roots containing β_i in their support also contain some β_l with $m_l < m_i$.

(3) If G is of type G_2 in characteristic 2, the only case where a parabolic not of standard type (with maximal reduced part) appears is when

$$P = P^{\alpha_2} \cap {}_mGP^{\alpha_1},$$

for which we get one exotic stabiliser, namely

$$Q^1 = {}_{m-1}LP^{\alpha_1}.$$

PROOF. **(1)** Assume G is not of type G_2 and does not satisfy the edge hypothesis. Then by Theorem 3.3.2, the only parabolic subgroup scheme with reduced part P^{β_i} and whose intersection with $U_{-\beta_i}$ has height m_i is

$${}_{m_i}GP^{\beta_i},$$

so the latter necessarily coincides with Q^i.

(2) Assume G satisfies the edge hypothesis. By Lemma 4.2.1 together with Theorem 3.3.2, Q^i is obtained from its reduced part either by fattening with ${}_{m_i}G$ or with ${}_{m_i-1}N$. The second case can happen if and only if the equality

$$(P =) \quad {}_{m_i}GP^{\beta_i} \cap \left(\bigcap_{j \neq i} {}_{m_j}GP^{\beta_j} \right) = {}_{m_i-1}NP^{\beta_i} \cap \left(\bigcap_{j \neq i} {}_{m_j}GP^{\beta_j} \right) \tag{5.4.3}$$

is satisfied. Let φ and ψ be respectively the function associated to the left and the right hand term. If the support of γ does not contain β_i, then the whole root subgroup $U_{-\gamma}$ is contained in P^{β_i}, hence

$$\varphi(\gamma) = \psi(\gamma) = \infty.$$

Thus we can assume that β_i is in the support of γ.

If γ also contains some β_l in its support satisfying $m_l < m_i$, then the two functions still coincide on γ. Finally, assume that the support of γ only contains simple roots $\beta_l \in \Delta \backslash I$ with $m_l \geq m_i$. If γ is short, then

$${}_{m_i-1}N \cap U_{-\gamma}$$

has height equal to m_i, hence $\varphi(\gamma) = \psi(\gamma)$ once again. On the other hand, if γ is long then the intersection above has height $m_i - 1$: this is the only case where the equality (5.4.3) cannot hold, because

$$\varphi(\gamma) = m_i \quad \text{while} \quad \psi(\gamma) = m_i - 1.$$

Summarizing, the parabolic subgroup Q^i is *not* of standard type, and is in particular equal to

$${}_{m_i-1}NP^{\beta_i},$$

if and only if all positive long roots containing β_i in their support also contain some β_l with $m_l < m_i$.

(3) The last case to look at is when G is of type G_2, the characteristic is $p = 2$ and the

Picard rank of G/P is equal to 2.
Let us assume that

$$P = P^{\alpha_2} \cap {}_m G P^{\alpha_1}$$

for some positive m; then by Lemma 4.2.1 the intersection $Q^1 \cap U_{-\alpha_1}$ has height m, which means that the root subspace associated to $-\alpha_1$ is contained in $\operatorname{Lie} Q^1$. Hence by Lemma 4.3.12, the latter must contain the Lie subalgebra $\mathfrak{l}$. This fact, together with the classification of parabolic subgroups with reduced part P^{α_1}, given in Proposition 4.3.13, implies that ${}_{m-1}LP^{\alpha_1}$ is contained in Q^1. Moreover, the equality

$$P = P^{\alpha_2} \cap {}_{m-1}LP^{\alpha_1}$$

holds, because intersecting both sides with $U_{-\gamma}$ for any positive root $\gamma \neq \alpha_1$ gives a trivial intersection. This shows that

$$Q^1 = {}_{m-1}LP^{\alpha_1}.$$

Thus, in this case we obtain a parabolic which is not of quasi-standard type.
On the other hand, if

$$P = P^{\alpha_1} \cap {}_m G P^{\alpha_2}$$

for some m, then the intersection $Q^2 \cap U_{-\alpha_2}$ has height m, hence by Theorem 3.3.2 we have necessarily that

$$Q^2 = {}_m G P^{\alpha_2}$$

is standard. □

Example 5.4.6. Assume that the rank r is at least equal to two, that the integer m_r is nonzero and that the simple root β_r is long. Then $Q^r = {}_{m_r}GP^{\beta_r}$ is of standard type.

Example 5.4.7. A few parabolic subgroups in rank two, with

$$P_{\text{red}} = P^\alpha \cap P^\beta,$$

are listed below: in particular, these examples underline the importance of paying attention to the duality between the groups of type B_n and C_n.

P	type	Q^α	Q^β
$P^{\alpha_{n-1}} \cap {}_m G P^{\alpha_n}$	B_n	$Q^{n-1} = P^{\alpha_{n-1}}$	$Q^n = {}_{m-1}NP^{\alpha_n}$
$P^{\alpha_1} \cap {}_m G P^{\alpha_n}$	C_n	$Q^1 = P^{\alpha_1}$	$Q^n = {}_m G P^{\alpha_n}$
$P^{\alpha_n} \cap {}_m G P^{\alpha_1}$	C_n	$Q^n = P^{\alpha_n}$	$Q^1 = {}_{m-1}NP^{\alpha_1}$
$P^{\alpha_n} \cap {}_m G P^{\alpha_1}$	B_n	$Q^n = P^{\alpha_n}$	$Q^1 = {}_m G P^{\alpha_1}$

5.4.2. Existence of smooth contractions. We address the question of whether a rational projective homogeneous variety admits a smooth contraction. We obtain in Proposition 5.4.14 a structure result, saying that such a variety can be obtained by iterated Zariski locally trivial fibrations whose fibers are flag varieties of Picard rank one. Let us start by considering morphisms of maximal relative Picard rank.

Lemma 5.4.8. *Let X be a rational projective homogeneous variety. Then there is a semisimple, simply connected group G and a parabolic subgroup P of G, satisfying*

$$X = G/P,$$

such that for each simple factor of G, its intersection with P does not contain the kernel of any isogeny with no central factor.

PROOF. Let us write the variety X as

$$X = G'/P'$$

for some semisimple and simply connected group G' and a parabolic subgroup P' of G'. Let H be the subgroup of G' generated by all normal noncentral subgroups of height one of G' which are contained in P'. Then we can write

$$X = (G'/H)/(P'/H) =: G/P.$$

The group G is still simply connected and the parabolic subgroup P is as wanted. □

The question of the existence of a smooth contraction is now related to finding some simple root α, not belonging to the Levi subgroup of the reduced part of P, such that the stabiliser Q^α is reduced (hence, smooth).

Remark 5.4.9. If P' is a parabolic subgroup of quasi-standard type of G', then Lemma 5.4.8 can be illustrated in a simpler way as follows. Let

$$P' = \bigcap_{i=1}^{r} (\ker \xi_i) P^{\beta_i},$$

where r is the Picard rank of X and the ξ_i are isogenies with no central factor, uniquely determined as in Theorem 3.3.2. Moreover, let us order them such that

$$\ker \xi_1 \subseteq \ldots \subseteq \ker \xi_r,$$

which can be done thanks to Remark 2.5.13. Then the subgroup H of G generated by all normal noncentral subgroups of height one contained in P is

$$H = \ker \xi_1.$$

Proposition 5.4.10. *Let G be simply connected and $X = G/P$ be such that P contains no kernel of isogenies with no central factor.*
Then there is a simple root α not belonging to the Levi subgroup of P_{red} such that

$$Q^\alpha = P^\alpha$$

if and only if P is of quasi-standard type.

PROOF. **(1)**: Let us assume that P is of quasi-standard type (which we recall is always the case except for a group of type G_2 in characteristic two). By Remark 5.4.9, we can write

$$P = P^\alpha \cap (\ker \xi) P'$$

for some simple root α and an isogeny ξ with no central factor. Then Q^α is equal to P^α, which in particular means that

$$f_\alpha \colon X = G/P \longrightarrow G/P^\alpha$$

is a smooth contraction.
(2): Let us assume that G is of type G_2, that $p = 2$ and that P is not of quasi-standard

type. Since by assumption P does not contain the Frobenius kernel of G, it must be of the form

$$P = P_{\mathfrak{h}} \cap {}_sGP^{\alpha_2} \quad \text{or} \quad P = P_{\mathfrak{l}} \cap {}_sGP^{\alpha_2}$$

for some non-negative integer s. In both cases, we have that the height of the intersection of Q^2 and $U_{-\alpha_2}$ is equal to s. Hence

$$Q^2 = {}_sGP^{\alpha_2}$$

is smooth if and only if s is zero: this cannot happen because

$$P = P_{\mathfrak{h}} \cap P^{\alpha_2} = P^{\alpha_1} \cap P^{\alpha_2} = B \quad \text{and} \quad P = P_{\mathfrak{l}} \cap P^{\alpha_2} = {}_1GP^{\alpha_1} \cap P^{\alpha_2}$$

are both of standard type, which contradicts our assumption.
Thus we have $s \geq 1$. Let us consider the subgroup Q^1: in the first case, the height of the intersection of Q^1 and $U_{-2\alpha_1-\alpha_2}$ is equal to one, which implies that $\operatorname{Lie} Q^1$ contains $\mathfrak{h}$ and finally that

$$Q^1 = P_{\mathfrak{h}}$$

is nonreduced. Analogously, in the second case the height of the intersection of Q^1 and $U_{-\alpha_1}$ is equal to one, which implies that $\operatorname{Lie} Q^1$ contains $\mathfrak{l}$ and finally that

$$Q^1 = P_{\mathfrak{l}}$$

is again non-reduced. □

Corollary 5.4.11. *Over an algebraically closed field of characteristic $p = 2$, the isomorphism classes of G_2-homogeneous varieties with Picard group of rank 2 are in one-to-one correspondence with the following parabolic subgroups:*

(a) $P^{\alpha_1} \cap P^{\alpha_2} = B$;
(b) ${}_mGP^{\alpha_1} \cap P^{\alpha_2}$, *for* $m \geq 1$;
(c) $P^{\alpha_1} \cap {}_mGP^{\alpha}$, *for* $m \geq 1$;
(d) $P_{\mathfrak{h}} \cap {}_mGP^{\alpha_2}$, *for* $m \geq 1$;
(e) $P_{\mathfrak{l}} \cap {}_mGP^{\alpha_2}$, *for* $m \geq 1$.

Remark 5.4.12. In the above list, (a), (b) and (c) are of quasi-standard type, while (d) and (e) are not. This leads to different geometric properties of the homogeneous spaces: for instance, the variety with stabiliser (a) has two smooth contractions, those in cases (b) and (c) have one, while those in cases (d) and (e) have none; see Proposition 5.4.14 below.

Before proving Corollary 5.4.11, let us recall that, for the notion of automorphism group, we use the notation introduced in Remark 2.3.1.

Proof. (of Corollary 5.4.11). Let us write

$$X = G/P = G'/P',$$

where G is of type G_2, P is a parabolic subgroup with reduced part equal to the Borel B, and G' is an adjoint semisimple group, such that both actions on X are faithful. By Theorem 5.2.3 applied to the G'-action, the contraction

$$f_{\alpha_2} \colon X \longrightarrow G/Q^{\alpha_2} \simeq G/P^{\alpha_2} =: Y$$

induces an action of G' onto Y. By [Dem, Theorem 1], the reduced automorphism group

$$\underline{\mathrm{Aut}}^0_Y$$

of Y is G; in particular, the only semisimple group which can act faithfully and transitively on Y is the group G, which implies $G' = G$. We can thus deduce that P and P' are conjugated in G, because the group of type G_2 has no outer automorphisms. Thanks to our classification of parabolic subgroups, we get the desired list above. □

We now associate to any homogeneous space X - with stabiliser of quasi-standard type - a canonical fibration which, whenever the stabiliser of X is non-reduced, is realised as the smooth contraction with minimal relative Picard rank.
Here we focus on isomorphism classes of varieties rather than on conjugacy classes of parabolic subgroups, so we start by applying Lemma 5.4.8. Then we repeat this construction to build from X a finite sequence of Zariski locally trivial contractions, whose fibers are generalised flag varieties of Picard rank one. Let us start by noticing that the quotient morphism

$$G \longrightarrow G/P$$

is Zariski locally trivial (and in particular, smooth) if and only if the parabolic subgroup is reduced.

Remark 5.4.13. Let

$$X = G/P$$

with P a parabolic subgroup of quasi-standard, such that for each simple factor of G, its intersection with P does not contain the kernel of any isogeny with no central factor. Then there is a unique minimal reduced parabolic subgroup of G containing P, given by

$$P^{\mathrm{sm}} = \bigcap \{P^\alpha \colon \alpha \in \Delta \backslash I \ \text{ and } \ Q^\alpha = P^\alpha\}.$$

Moreover, P can be written in a unique way as

$$P = P^{\mathrm{sm}} \cap (\ker \xi) P', \tag{5.4.4}$$

where ξ is an isogeny with no central factor and P' is a parabolic subgroup with reduced part

$$P'_{\mathrm{red}} = \bigcap \{P^\alpha \colon \alpha \in \Delta \backslash I \ \text{ and } \ Q^\alpha \neq P^\alpha\},$$

such that P' does not contain the kernel of any isogeny with no central factor.

Proposition 5.4.14. *Let X be a homogeneous projective variety of Picard rank $r \geq 2$ whose automorphism group has no G_2 factor if the characteristic is 2.*
There is a finite sequence of Zariski locally trivial contraction morphisms

$$g_s \colon X_s \longrightarrow Y_s, \quad 1 \leq s \leq r$$

such that $X_1 = X$, each Y_s is a rational projective homogeneous variety of Picard rank one and X_{s+1} is the fiber of g_s.

PROOF. Let us write

$$X = G/P$$

with P a parabolic subgroup of quasi-standard type; thanks to Lemma 5.4.8 and Proposition 5.4.10, assume that there is some simple root α such that Q^α is smooth. Then the contraction morphism

$$f_\alpha \colon X \longrightarrow G/P^\alpha =: Y_1$$

is a Zariski locally trivial contraction, with smooth, connected and projective fiber equal to

$$X' := P^\alpha/P,$$

The idea is now to replace X by X'; in order to do this, let us consider the image

$$G^\alpha := \operatorname{im}(P^\alpha \longrightarrow \underline{\operatorname{Aut}}_{X_1})$$

which is a semisimple adjoint group. Let $\overline{P}$ be the image of P, then let G' and P' be respectively the simply connected cover of G^α and the preimage of $\overline{P}$ in G'. Then we have

$$X' = G^\alpha/\overline{P} = G'/P'.$$

This allows to start again by replacing the variety X by X'; since the Picard rank of the fiber decreases by one at each step, this process terminates in r steps. □

Let us mention that if $p > 3$ then for any parabolic subgroup P the Chow motives of G/P and of G/P_{red} are isomorphic, as proven in [Sri, Theorem 1.3]; we wonder whether this result could be recovered from Proposition 5.4.14.

5.4.3. Anti-canonical bundle and Frobenius splitting. Over an algebraically closed field of characteristic zero, the anti-canonical bundle on G/P is always globally generated, and every G/P is a Fano variety. Moreover, there are only a finite number of isomorphism classes of (projective, rational) homogeneous varieties with a fixed dimensions. All of these statements prove false in positive characteristics.

Example 5.4.15. Let us consider the following family of incidence varieties in the product of two projective planes:

$$X_m := \{x_0^{p^m} y_0 + x_1^{p^m} y_1 + x_2^{p^m} y_2 = 0\} \subset \mathbf{P}^2 \times \mathbf{P}^2,$$

where $m \geq 0$. The group SL_3 acts on the first factor with a twisted action via an m-th iterated Frobenius morphism, and with the standard action on the second factor, thus preserving X_m. These form an infinite family of non-isomorphic rational projective homogeneous spaces of dimension 3, which are not Fano if $p^m > 3$.

Nevertheless, the following finiteness property still holds: only a finite number of homogeneous spaces of fixed dimension are Fano varieties. The idea behind this result is that, except for a finite number of cases, there is at least one *incidence relation* in the embedding

$$f \colon X = G/P \longhookrightarrow \prod_{\alpha \in \Delta \setminus I} G/Q^\alpha$$

which is *twisted too much* via the kernel of an isogeny with no central factor.

For a simply connected semisimple algebraic group G and a parabolic subgroup P, any line bundle G/P admits a unique G-linearisation. Then one can associate to a line bundle L a unique character λ, given by restricting the G-action to the fiber over the base point.

Lemma 5.4.16. *A line bundle L on G/P with associated character λ is ample if and only if*

$$(\lambda, \alpha) > 0 \quad \textit{for all } \alpha \in \Delta \backslash I.$$

PROOF. If P is reduced, this holds by [**Jan**, II.8.5, II.4.4]. In general, let us consider the finite morphism

$$\sigma\colon G/P_{\mathrm{red}} \longrightarrow G/P.$$

The character associated to σ^*L is the restriction of λ to P_{red}. This, together with the fact that L is ample if and only if σ^*L is ample, allows us to conclude. □

Lemma 5.4.17. *The character associated to the anticanonical bundle of G/P is given by*

$$\chi = \sum_{\gamma \in \Phi^+ \backslash \Phi_I} p^{\varphi(\gamma)} \gamma,$$

where φ is the associated numerical function to the parabolic subgroup P.

PROOF. See [**Lau1**, Proposition 3.1]: the key point is that in the proof there is no assumption on the characteristic. □

THEOREM 5.4.18. *Let $n \geq 1$ be a fixed integer.*
There are a finite number of isomorphism classes of rational projective homogeneous varieties of dimension n whose character associated to the anti-canonical bundle is dominant *(equivalently, whose anti-canonical bundle is globally generated).*

PROOF. Such varieties are all of the form

$$X = G/P$$

where G is semisimple, simply connected and P is a parabolic subgroup.
Up to replacing the adjoint quotient of G with its image into the automorphism group of X, we can assume that G acts on X with a finite kernel. In particular, the same is true for a maximal torus $T \subset G$; this implies that the stabiliser

$$\mathrm{Stab}_T(x)$$

is finite for a general point $x \in X$ and thus that

$$n = \dim X \geq \dim T = \operatorname{rank} G.$$

Summarizing, we have just proved that the rank of G is bounded by the dimension of X, which is fixed and equal to n. Any such G is semisimple and simply connected, thus product of simple factors; there are finitely many isomorphism classes of such groups with rank less or equal than n, thus there are finitely many possibilities for G.
Next, we fix a Borel subgroup B: we can assume that P contains B and moreover, that P does not contain the kernel of any isogeny with no central factor.
Step 1: if the reduced part of P is maximal, then by Theorem 3.3.2, X is either isomorphic to a flag variety with reduced stabiliser, whose character associated to the anti-canonical

bundle is in particular dominant, or to the exotic variety with stabiliser P_{l} in type G_2 (see Proposition 4.3.13 for this case): these are a finite number of non-isomorphic varieties, so we can exclude them.

Step 2: Let us assume that P is quasi-standard: then we have

$$\Delta\backslash I = \{\beta_1, \dots, \beta_r\}, \quad \text{where } P_{\text{red}} = P_I.$$

Up to re-arranging $\beta_1, \dots, \beta_r$, we can find isogenies $\xi_2, \dots, \xi_r$ with source G, uniquely determined by the conditions

$$\langle P, P^{\beta_i} \rangle = (\ker \xi_i) P^{\beta_i} \quad \text{and} \quad \ker \xi_2 \subseteq \ldots \subseteq \ker \xi_r,$$

such that we can write P as

$$P = P^{\beta_1} \cap (\ker \xi_2) P^{\beta_2} \cap \ldots \cap (\ker \xi_r) P^{\beta_r}.$$

We can assume that $r \geq 2$, because the case of $r = 1$ has been treated in Step 1. Let us consider a positive integer m, big enough such that it satisfies

$$p^m > H := \frac{\max_{\alpha \in \Delta} \sum_{\alpha \in \text{Supp}(\gamma)} |(\gamma, \alpha)|}{\min_{\alpha \in \Delta, (\gamma, \alpha) < 0} |(\gamma, \alpha)|}. \tag{5.4.5}$$

By Remark 2.5.13, if we exclude a *finite* number of cases, we can assume that there is some $i < r$ and some $m_0 \in \mathbf{N}$ such that

$$\ker \xi_2 \subseteq \ldots \subseteq \ker \xi_i \subseteq {}_{m_0}G \quad \text{and} \quad {}_{m_0+m}G \subseteq \ker \xi_{i+1} \subseteq \ldots \subseteq \ker \xi_r. \tag{5.4.6}$$

Claim: if P satisfies (5.4.6), then the character χ associated to the anti-canonical bundle on X is not dominant.

To prove the claim, let us write $\Phi^+\backslash\Phi_I$ as the disjoint union of $\Psi_{\geq}$ and $\Psi_{\leq}$, defined as:

$$\Psi_{\geq} := \{\gamma \in \Phi^+\backslash\Phi_I,\ \text{Supp}(\gamma) \cap \{\beta_1, \dots, \beta_i\} = \emptyset\};$$
$$\Psi_{\leq} := \{\gamma \in \Phi^+\backslash\Phi_I,\ \exists j \leq i \text{ such that } \beta_j \in \text{Supp}(\gamma)\}.$$

The condition (5.4.6) implies that

$$\varphi(\gamma) \leq m_0 \text{ for } \gamma \in \Psi_{\leq}, \quad \varphi(\gamma) \geq m_0 + m \text{ for } \gamma \in \Psi_{\geq}.$$

Let us fix some β_l with $l \leq i$ and some $\delta \in \Psi_{\geq}$ such that

$$(\delta, \beta_l) = -s \quad \text{for some } s > 0,$$

which exists thanks to Lemma 5.4.19 below. Let us notice that $(\gamma, \beta_l) > 0$ implies that $\beta_l \in \text{Supp}(\gamma)$, hence

$$(\gamma, \beta_l) \leq 0 \quad \text{for all } \gamma \in \Psi_{\geq}.$$

On the other hand, let us write $\Psi_{\leq}$ as the disjoint union of

$$\Psi_{\leq}^- := \{\gamma \in \Psi_{\leq},\ (\gamma, \beta_l) \leq 0\};$$
$$\Psi_{\leq}^+ := \{\gamma \in \Psi_{\leq},\ (\gamma, \beta_l) > 0\} \subset \{\gamma \in \Phi^+,\ \beta_l \in \text{Supp}(\gamma)\}.$$

In order to conclude, it is enough to show that

$$(\chi, \beta_l) < 0, \tag{5.4.7}$$

By Lemma 5.4.17, we get

$$(\chi,\beta_l)=\sum_{\gamma\in\Psi_{\leq}}p^{\varphi(\gamma)}(\gamma,\beta_l)+\sum_{\gamma\in\Psi_{\geq}}p^{\varphi(\gamma)}(\gamma,\beta_l)\leq\sum_{\gamma\in\Psi_{\leq}^{+}}p^{\varphi(\gamma)}(\gamma,\beta_l)+\sum_{\gamma\in\Psi_{\geq}}p^{\varphi(\gamma)}(\gamma,\beta_l)$$

$$\leq p^{m_0}\sum_{\gamma\in\Psi_{\leq}^{+}}(\gamma,\beta_l)+p^{m_0+m}\sum_{\gamma\in\Psi_{\geq}}(\gamma,\beta_l)\leq p^{m_0}\left(\sum_{\gamma\in\Psi_{\leq}^{+}}(\gamma,\beta_l)-p^m s\right)$$

Let us consider the integer

$$N:=\sum_{\gamma\in\Psi_{\leq}^{+}}(\gamma,\beta_l)\leq\sum_{\gamma\in\Phi^{+},\,\beta_l\in\mathrm{Supp}(\gamma)}|(\gamma,\beta_l)|$$

Then, by the assumption (5.4.5) we have that

$$p^m>H\geq N/s,$$

hence

$$(\chi,\beta_l)\leq p^{m_0}(N-p^m s)<0$$

and we have proved (5.4.7).

Step 3: The last case to treat is the one of a group G of type G_2 in characteristic $p=2$ and of a parabolic subgroup which is not standard. By Step 1 above, we can assume that the reduced part of P is the Borel B. By Corollary 5.4.11, the variety X is isomorphic to exactly one G/P with P belonging to the following list:

$$B,\quad {}_mGP^{\alpha_1}\cap P^{\alpha_2},\quad P^{\alpha_1}\cap{}_mGP^{\alpha},\quad P_{\mathfrak{h}}\cap{}_mGP^{\alpha_2},\quad P_{\mathfrak{l}}\cap{}_mGP^{\alpha_2},\ \text{for } m\geq 1.$$

Clearly, the character of the anti-canonical bundle on the variety G/B is dominant. Next, let us assume that P is nonreduced and that $m\geq 2$, thus excluding a finite number of varieties: under this assumption, we claim that the character χ is not dominant. In order to make computations, let us recall that we have

$$(\alpha_1,\alpha_1)=2,\quad(\alpha_1,\alpha_2)=-3,\quad(\alpha_2,\alpha_2)=6.$$

Let us proceed by a case-by-case analysis, using the canonical bundle formula of Lemma 5.4.17.

- If $P={}_mGP^{\alpha_1}\cap P^{\alpha_2}$, then

$$\varphi(\alpha_1)=m\quad\text{and}\quad\varphi=0\text{ on }\Phi^{+}\backslash\{\alpha_1\}.$$

Thus

$$(\chi,\alpha_2)=((2^m+9)\alpha_1+6\alpha_2,\alpha_2)=9-3\cdot 2^m<0.$$

- If $P=P^{\alpha_1}\cap{}_mGP^{\alpha_2}$, then

$$\varphi(\alpha_2)=m\quad\text{and}\quad\varphi=0\text{ on }\Phi^{+}\backslash\{\alpha_2\}.$$

Thus

$$(\chi,\alpha_1)=(10\alpha_1+(5+2^m)\alpha_2,\alpha_1)=5-3\cdot 2^m<0.$$

- If $P=P_{\mathfrak{h}}\cap{}_mGP^{\alpha_2}$, then

$$\varphi(\alpha_2)=m,\quad\varphi(2\alpha_1+\alpha_2)=1\quad\text{and}\quad\varphi=0\text{ on }\Phi^{+}\backslash\{\alpha_2,2\alpha_1+\alpha_2\}.$$

Thus

$$(\chi, \alpha_1) = ((4+8)\alpha_1 + (2^m+6)\alpha_2, \alpha_1) = 6 - 3 \cdot 2^m < 0. \tag{5.4.8}$$

- If $P = P_{\mathfrak{l}} \cap {}_m G P^{\alpha_2}$, then

$$\varphi(\alpha_2) = m, \quad \varphi(\alpha_1) = \varphi(\alpha_1 + \alpha_2) = 1 \quad \text{and} \quad \varphi = 0 \text{ on } \Phi^+ \backslash \{\alpha_2, \alpha_1, \alpha_1 + \alpha_2\}.$$

Then we get the exact same computation as in (5.4.8) and we can conclude. □

Lemma 5.4.19. *Let $1 \leq i < r$ and consider a partition of simple roots as follows:*

$$\Delta \backslash I = \{\beta_1 \dots \beta_i\} \cup \{\beta_{i+1} \dots \beta_r\}.$$

Then there is some $l \leq i$ and some $\delta \in \Phi^+ \backslash \Phi_I$ such that

$$\mathrm{Supp}(\delta) \cap \{\beta_1, \dots, \beta_i\} = \emptyset \quad \textit{and} \quad (\delta, \beta_l) < 0.$$

Proof. Let us fix

$$\nu \in \{\beta_1, \dots, \beta_i\} \quad \text{and} \quad \mu \in \{\beta_{i+1}, \dots, \beta_r\},$$

such that the couple (ν, μ) realises the minimum of the distances between the corresponding nodes in the Dynkin diagram. In particular, there is a segment of minimal length of nodes $J \subset \Delta$, having as extremes the nodes ν and μ; either these two are adjacent, or the nodes in between them are all simple roots belonging to I. Let us consider the interior

$$K := J \backslash \{\nu, \mu\} \subset I.$$

Then, we can set

$$\beta_l := \nu \quad \text{and} \quad \delta := \mu + \sum_{\alpha \in K} \alpha.$$

Since either μ or some root of K is adjacent to ν, and moreover ν is not in the support of δ, we can conclude that (δ, β_l) is strictly negative. □

Corollary 5.4.20. *Let $n \geq 1$ be a fixed integer.*
There are a finite number of isomorphism classes of projective homogeneous varieties of dimension n which are Fano.

Proof. This is an immediate consequence of Lemma 5.4.16 above, together with Theorem 5.4.18. □

Let us conclude by mentioning another direct geometric consequence of the above Theorem 5.4.18. Namely, we show that finitely many homogeneous spaces of a fixed dimension are Frobenius split. Let us first recall what this splitting property is and let us mention some previous results.

Definition 5.4.21. A variety (over an algebraically closed field of characteristic $p > 0$) is said to be *Frobenius split* if the morphism

$$\mathcal{O}_{X^{(1)}} \longrightarrow (F_X)_* \mathcal{O}_X$$

splits as a morphism of $\mathcal{O}_{X^{(1)}}$-modules, where

$$F_X \colon X \longrightarrow X^{(1)}$$

is the relative Frobenius morphism of X.

This property gives important information on the geometry. For instance, if a variety X is Frobenius split, then vanishing for ample line bundles holds on X. In particular, partial flag varieties (namely, homogeneous varieties with reduced stabilizers) are Frobenius split.

Lauritzen, in [**Lau1**, Theorem 5.2], shows the following: if p is strictly greater than the Coxeter number of G, then the following are equivalent:

- G/P is Frobenius split;
- there is some integer m such that $P = {}_mGP_{\mathrm{red}}$.

Clearly, this is not sufficient for our purposes. In his result, the characteristic is assumed to be large enough, so that the only parabolics that the statement deals with are of standard type. However, what is of interest for us is the following result, which is a particular case of [**Lau1**, Lemma 1.2] and which needs no hypothesis on the characteristic. Let us denote as K_X the canonical bundle of a variety X. We choose to use additive notation on line bundles here, in order to reflect the corresponding additive notation of the character group of P.

Lemma 5.4.22. *If the line bundle*

$$-(p-1)K_X$$

has no global sections, then the variety X is not Frobenius split.

Corollary 5.4.23. *Let $n \geq 1$ be a fixed integer.*
There are a finite number of isomorphism classes of projective homogeneous varieties of dimension n which are Frobenius split.

Proof. A line bundle on a homogeneous variety G/P admits global sections if and only if the corresponding character (in the character group of P) is dominant. By Theorem 5.4.18, we have the following: for all but finitely many homogeneous varieties of dimension n, the anti-canonical bundle has no global sections. This yieds that $-(p-1)K_X$ also has no global sections; hence by Lemma 5.4.22 we are done. □

CHAPTER 6

Appendix

ABSTRACT. We summarize here a description of the group of type G_2 as automorphism group of an octonion algebra, which holds in any characteristic. We then specialize to characteristic two which is the interesting one for our purposes.

6.1. The embedding of G_2 into SO_7

Let G be the simple group of type G_2, over an algebraically closed field k of characteristic $p > 0$. The group G can be viewed - as illustrated in [**SV**], from which we keep most of the notation - as the automorphism group of an octonion algebra. The latter is the algebra

$$\mathbb{O} = \{(u, v) \colon u, v \text{ are } 2 \times 2 \text{ matrices}\},$$

with basis

$$e_{11} = \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right), \qquad e_{12} = \left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right),$$

$$e_{21} = \left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right), \qquad e_{22} = \left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right),$$

$$f_{11} = \left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right), \qquad f_{12} = \left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right),$$

$$f_{21} = \left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right), \qquad f_{22} = \left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right),$$

unit $e = (1, 0) = e_{11} + e_{22}$, and which is equipped with a norm

$$q(u, v) = \det(u) - \det(v).$$

Let us write here for reference a table of products of the basis vectors :

(6.1.1)

$\diagdown$	e_{11}	e_{21}	e_{12}	e_{22}	f_{11}	f_{21}	f_{12}	f_{22}
e_{11}	e_{11}	0	e_{12}	0	f_{11}	f_{21}	0	0
e_{21}	e_{21}	0	e_{22}	0	0	0	f_{11}	f_{21}
e_{12}	0	e_{11}	0	e_{12}	f_{12}	f_{22}	0	0
e_{22}	0	e_{21}	0	e_{22}	0	0	f_{12}	f_{22}
f_{11}	0	0	$-f_{12}$	f_{11}	0	$-e_{21}$	0	e_{11}
f_{21}	0	0	$-f_{22}$	f_{21}	e_{21}	0	$-e_{11}$	0
f_{12}	f_{12}	$-f_{11}$	0	0	0	$-e_{22}$	0	e_{12}
f_{22}	f_{22}	$-f_{21}$	0	0	e_{22}	0	$-e_{12}$	0

An embedding of the group G into SO_7 - which gives an irreducible representation in all characteristics but two - can be seen as follows: let us consider the G-action on the vector space

$$V := e^{\perp} = \{(u,v)\colon \det(1+u) - \det(u) = 1\} = \{(u,v)\colon u_{11}+u_{22}=0\}. \tag{6.1.2}$$

By [**SV**, Lemma 2.3.1], a maximal torus of G - with respect to the basis

$$(e_{12}, e_{21}, f_{11}, e_{11}-e_{22}, -f_{12}, f_{21}, f_{22})$$

of V - acts on V as

$$\mathbf{G}_m{}^2 \ni (\xi,\eta) \longmapsto \mathrm{diag}(\xi\eta, \xi^{-1}\eta^{-1}, \eta^{-1}, 1, \xi, \xi^{-1}, \eta) \in \mathrm{GL}_7$$

Let us re-parameterize it with $\xi = a$, $\eta = ab$, this gives the torus

$$\mathbf{G}_m{}^2 \ni (a,b) \longmapsto \mathrm{diag}(a^2b, a^{-2}b^{-1}, a^{-1}b^{-1}, 1, a, a^{-1}, ab) =: t \in \mathrm{GL}_7,$$

and the basis of simple roots we fix is $\alpha_1(t) := a$ and $\alpha_2(t) := b$. Such a torus acts on V with the following weight spaces :

$$V_0 = k(e_{11}-e_{22}), V_{\alpha_1} = kf_{12},\ V_{-\alpha_1} = kf_{21},\ V_{\alpha_1+\alpha_2} = kf_{22},$$
$$V_{-\alpha_1-\alpha_2} = kf_{11},\ V_{2\alpha_1+\alpha_2} = ke_{12},\ V_{-2\alpha_1-\alpha_2} = ke_{21},$$

which correspond to 0 and the short roots. Re-arranging V as

$$V = k(-f_{12}) \oplus kf_{11} \oplus ke_{12} \oplus k(e_{11}-e_{22}) \oplus ke_{21} \oplus kf_{22} \oplus kf_{21} \tag{6.1.3}$$

gives the maximal torus T

$$\mathbf{G}_m{}^2 \ni (a,b) \longmapsto \mathrm{diag}(a, a^{-1}b^{-1}, a^2b, 1, a^{-2}b^{-1}, ab, a^{-1}) = t \in T \subset \mathrm{GL}_7\,. \tag{6.1.4}$$

This way, T can be identified with the maximal torus in [**Hei**, page 13]: in his description of the embedding $G \subset \mathrm{GL}_7$, the group G is generated by the two following copies of GL_2,

$$\theta_1\colon A \longmapsto \begin{pmatrix} A & & \\ & \mathrm{Sym}^2(A)\det A^{-1} & \\ & & A \end{pmatrix} \quad\text{and}\quad \theta_2\colon B \longmapsto \begin{pmatrix} \det B^{-1} & & & & \\ & \widetilde{B} & & & \\ & & 1 & & \\ & & & B & \\ & & & & \det B \end{pmatrix},$$

where

$$\widetilde{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} {}^t A^{-1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

However, in characteristic $p = 2$, due to the fact that $e \in V$ and that G acts on the quotient $W = V/ke$, these become the following two copies embedded in $\mathrm{GL}(W) = \mathrm{GL}_6$:

$$\theta_1\colon A \longmapsto \begin{pmatrix} A & & \\ & A^{(1)}\det A^{-1} & \\ & & A \end{pmatrix} \quad\text{and}\quad \theta_2\colon B \longmapsto \begin{pmatrix} \det B^{-1} & & & \\ & B & & \\ & & B & \\ & & & \det B \end{pmatrix},$$

where $A^{(1)}$ denotes the Frobenius twist applied to A.

Lemma 6.1.1. *The subgroups $\theta_1(\mathrm{GL}_2)$ and $\theta_2(\mathrm{GL}_2)$ have root system with positive root respectively $\beta_1 := 2\alpha_1+\alpha_2$ and $\beta_2 := -3\alpha_1-2\alpha_2$.*

PROOF. See the computation of the root homomorphisms associated respectively to β_1 and β_2, done in Remark 6.2.1 below: these are respectively the intersection of $\theta_1(\mathrm{GL}_2)$ and $\theta_2(\mathrm{GL}_2)$ with the upper triangular matrices of GL_7. □

Let us remark that $\{\beta_1, \beta_2\}$ is indeed a basis for the root system of type G_2, with corresponding set of positive roots being

$$-3\alpha_1 - 2\alpha_2,\ \alpha_1 - \alpha_2,\ -\alpha_2,\ \alpha_2,\ 3\alpha_1 + \alpha_2,\ 2\alpha_1 + \alpha_2$$

and with Borel subgroup given by the intersection of G with the upper triangular matrices in GL_7.

6.2. Root subgroups

Let us move on to the explicit computation of some of the root subgroups in type G_2. As before, we will do everything considering the action on a 7-dimensional vector space - the orthogonal of the identity element of $\mathbb{O}$ - so that the computations hold in any characteristic, then at the end we will summarize what we get in characteristic 2. This latter part is fundamental in order to study the exotic parabolic subgroups $P_{\mathfrak{h}}$ and $P_{\mathfrak{l}}$, introduced in Definition 3.2.9.

Let us consider the group G acting on the vector space V arranged as in (6.1.3). Denoting as $x_0, \ldots, x_6$ the coordinates on V, the norm becomes

$$q(x) = -x_3^2 - x_2x_4 - x_1x_5 - x_0x_6, \tag{6.2.1}$$

while the maximal torus T given in (6.1.4) acts on V through this table of characters

(6.2.2)

1	a^2b	$a^{-1}b^{-1}$	a	a^3b	b^{-1}	a^2
$a^{-2}b^{-1}$	1	$a^{-3}b^{-2}$	$a^{-1}b^{-1}$	a	$a^{-2}b^{-2}$	b^{-1}
ab	a^3b^2	1	a^2b	a^4b^2	a	a^3b
a^{-1}	ab	$a^{-2}b^{-1}$	1	a^2b	$a^{-1}b^{-1}$	a
$a^{-3}b^{-1}$	a^{-1}	$a^{-4}b^{-2}$	$a^{-2}b^{-1}$	1	$a^{-3}b^{-2}$	$a^{-1}b^{-1}$
b	a^2b^2	a^{-1}	ab	a^3b^2	1	a^2b
a^{-2}	b	$a^{-3}b^{-1}$	a^{-1}	ab	$a^{-2}b^{-1}$	1

The idea is the following: we know that - for any root $\gamma \in \Phi$ - the root subgroup $U_\gamma \subset G$ is determined by being the unique subgroup of $\mathrm{GL}(V)$ (resp. $\mathrm{GL}(W)$ in characteristic 2), which is smooth connected unipotent, is acted on by T via the character γ, and whose elements are automorphisms of octonions. We will impose some of these necessary conditions - such as $u_\gamma(\lambda)$ being an isometry for any $\lambda \in \mathbf{G}_a$ - to determine the root homomorphism $u_\gamma \colon \mathbf{G}_a \longrightarrow U_\gamma$.

• First, let us consider the root α_1. By (6.2.2) and the condition for u_{α_1} to be a group homomorphism, there exist some constants $\eta_1, \ldots, \eta_5 \in k$ such that for any $\lambda \in \mathbf{G}_a$,

$u_{\alpha_1}(\lambda)$ acts on V as

$$\begin{pmatrix} 1 & 0 & 0 & \eta_1\lambda & 0 & 0 & \eta_5\lambda^2 \\ 0 & 1 & 0 & 0 & \eta_2\lambda & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & \eta_3\lambda & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & \eta_4\lambda \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Moreover, $u_{\alpha_1}(\lambda)$ being an isometry means, by (6.2.1), that

$$\begin{aligned} q(x) = q(u_{\alpha_1}(\lambda)\cdot x) &= q(x_0+\eta_1\lambda x_3+\eta_5\lambda^2 x_6, x_1+\eta_2\lambda x_4, x_2+\eta_3\lambda x_5, x_3+\eta_4\lambda x_6, x_4, x_5, x_6) \\ &= q(x) + -(2\eta_4+\eta_1)\lambda x_3 x_6 + -(\eta_5+\eta_4^2)\lambda^2 x_6^2 - (\eta_3+\eta_2)\lambda x_4 x_5, \end{aligned}$$

hence $\eta_1 = -2\eta_4$, $\eta_5 = -\eta_4^2$ and $\eta_2 = -\eta_3$. This still leaves two independent parameters η_3 and η_4 instead of one, so let us also impose the condition of $u_{\alpha_1}(\lambda)$ respecting the product $e_{12}f_{21} = f_{22}$ - see (6.1.1) :

$$\begin{aligned} (u_{\alpha_1}(\lambda)\cdot e_{12})(u_{\alpha_1}(\lambda)\cdot f_{21}) &= u_{\alpha_1}(\lambda)\cdot(f_{22}) \\ e_{12}(\eta_4^2\lambda^2 f_{12} + \eta_4(e_{11}-e_{22}) + f_{21}) &= \eta_3\lambda e_{12} + f_{22} \\ -\eta_4\lambda e_{12} + f_{22} &= \eta_3\lambda e_{12} + f_{22}, \end{aligned}$$

implying $\eta_3 = -\eta_4$. Let us reparametrise the root homomorphism such that $\eta_4 = 1$: this, together with an analogous computation for $-\alpha_1$, gives the desired representations, of the form

$$u_{\alpha_1}: \lambda \mapsto \begin{pmatrix} 1 & 0 & 0 & -2\lambda & 0 & 0 & -\lambda^2 \\ 0 & 1 & 0 & 0 & \lambda & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -\lambda & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & \lambda \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad u_{-\alpha_1}: \lambda \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \lambda & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -\lambda & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 & 1 & 0 \\ -\lambda^2 & 0 & 0 & -2\lambda & 0 & 0 & 1 \end{pmatrix}.$$

• Let us consider the root α_2. By (6.2.2) and the condition for u_{α_2} to be a group homomorphism, there exist some constants η_1 and $\eta_2 \in k$ such that for any $\lambda \in \mathbf{G}_a$, $u_{\alpha_2}(\lambda)$ acts on V as

$$u_{\alpha_2}(\lambda)\cdot x = (x_0, x_1, x_2, x_3, x_4, \eta_1\lambda x_0 x_5, \eta_2\lambda x_1 + x_6).$$

Moreover, the isometry condition means that

$$\begin{aligned} q(x) = q(u_{\alpha_2}(\lambda)\cdot x) &= -x_3^2 - x_2x_4 - \eta_1\lambda x_0x_1 - x_1x_5 - \eta_2\lambda x_0x_1 - x_0x_6 \\ &= q(x) - (\eta_2+\eta_1)\lambda x_0 x_1, \end{aligned}$$

hence $\eta_1 = -\eta_2$. As before, we can conclude that the associated root subgroups are of the form

$$u_{\alpha_2}\colon \lambda \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -\lambda & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & \lambda & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad u_{-\alpha_2}\colon \lambda \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \lambda & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -\lambda \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

• Let us consider the root $2\alpha_1 + \alpha_2$. By (6.2.2) and the condition for $u_{2\alpha_1+\alpha_2}$ to be a group homomorphism, there exist some constants $\eta_1, \ldots, \eta_5 \in k$ such that for any $\lambda \in \mathbf{G}_a$, $u_{2\alpha_1+\alpha_2}(\lambda)$ acts on V as

$$\begin{pmatrix} 1 & \eta_1\lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \eta_2\lambda & \eta_5\lambda^2 & 0 & 0 \\ 0 & 0 & 0 & 1 & \eta_3\lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \eta_4\lambda \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Moreover, the isometry condition implies

$$\begin{aligned} q(u_{2\alpha_1+\alpha_2}(\lambda) \cdot x) &= q(x_0 + \eta_1\lambda x_1, x_1, x_2 + \eta_2\lambda x_3 + \eta_5\lambda^2 x_4, x_3 + \eta_3\lambda x_4, x_4, x_5 + \eta_4\lambda x_6, x_6) \\ &= q(x) - (\eta_1 + \eta_4)\lambda x_1 x_6 - (\eta_3^2 + \eta_5)\lambda^2 x_4^2 - (2\eta_3 + \eta_2)\lambda x_3 x_4 = q(x), \end{aligned}$$

hence $\eta_1 = -\eta_4$, $\eta_5 = -\eta_3^2$ and $\eta_2 = -2\eta_3$. This still leaves two independent parameters η_3 and η_4 instead of one, so let us also impose the condition of $u_{2\alpha_1+\alpha_2}(\lambda)$ respecting the product $f_{22}e_{21} = -f_{21}$:

$$\begin{aligned} (u_{2\alpha_1+\alpha_2}(\lambda) \cdot f_{22})(u_{2\alpha_1+\alpha_2}(\lambda) \cdot e_{21}) &= u_{2\alpha_1+\alpha_2}(\lambda) \cdot (-f_{21}) \\ f_{22}(-\eta_3^2\lambda^2 e_{12} + \eta_3\lambda(e_{11} - e_{22}) + e_{21}) &= -\eta_4\lambda f_{22} - f_{21} \\ \eta_3\lambda f_{22} - f_{21} &= -\eta_4\lambda f_{22} - f_{21}, \end{aligned}$$

implying $\eta_3 = -\eta_4$, so we can conclude that the associated root subgroups are of the form

$$u_{2\alpha_1+\alpha_2}\colon \lambda \mapsto \begin{pmatrix} 1 & -\lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2\lambda & -\lambda^2 & 0 & 0 \\ 0 & 0 & 0 & 1 & -\lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \lambda \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \; u_{-2\alpha_1-\alpha_2}\colon \lambda \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \lambda & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\lambda & 1 & 0 & 0 & 0 \\ 0 & 0 & -\lambda^2 & 2\lambda & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\lambda & 1 \end{pmatrix}.$$

• Let us consider the root $\alpha_1 + \alpha_2$. By (6.2.2) and the condition for $u_{\alpha_1+\alpha_2}$ to be a group homomorphism, there exist some constants $\eta_1, \ldots, \eta_5 \in k$ such that for any $\lambda \in \mathbf{G}_a$,

$u_{\alpha_1+\alpha_2}(\lambda)$ acts on V as

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \eta_1\lambda & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & \eta_2\lambda & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & \eta_5\lambda^2 & 0 & \eta_3\lambda & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \eta_4\lambda & 0 & 1 \end{pmatrix}.$$

Moreover, the isometry condition implies

$$\begin{aligned} q(x) =& q(u_{\alpha_1+\alpha_2}(\lambda) \cdot x) = q(x_0, x_1, \eta_1\lambda x_0 + x_2, \eta_2\lambda x_1 + x_3, x_4, \eta_5\lambda^2 x_1 + \eta_3\lambda x_3 + x_5, \eta_4\lambda x_4 + x_6) \\ =& q(x) - (2\eta_2 + \eta_3)\lambda x_1 x_3 - (\eta_4 + \eta_1)\lambda x_0 x_4 - (\eta_2^2 + \eta_5)\lambda^2 x_1^2, \end{aligned}$$

hence $\eta_3 = -2\eta_2$, $\eta_1 = -\eta_4$ and $\eta_5 = -\eta_2^2$. Reasoning as in the above cases, let us also impose the condition of $u_{\alpha_1+\alpha_2}(\lambda)$ respecting the product $f_{11}f_{21} = -e_{21}$:

$$\begin{aligned} (u_{\alpha_1+\alpha_2}(\lambda) \cdot f_{11})(u_{\alpha_1+\alpha_2}(\lambda) \cdot f_{21}) &= u_{\alpha_1+\alpha_2}(\lambda) \cdot (-e_{21}) \\ (f_{11} + \eta_2\lambda(e_{11} - e_{22}) - \eta_2^2\lambda^2 f_{22})f_{21} &= -e_{21} - \eta_4\lambda f_{21} \\ -e_{21} + \eta_2\lambda f_{21} &= -e_{21} - \eta_4\lambda f_{21}, \end{aligned}$$

implying $\eta_2 = -\eta_4$. Reparametrizing and doing an analogous computation for the negative root allows to conclude that the root subgroups are as follows :

$$u_{\alpha_1+\alpha_2} \colon \lambda \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -\lambda & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -\lambda & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -\lambda^2 & 0 & 2\lambda & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \lambda & 0 & 1 \end{pmatrix}, \quad u_{-\alpha_1-\alpha_2} \colon \lambda \mapsto \begin{pmatrix} 1 & 0 & \lambda & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2\lambda & 0 & -\lambda^2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -\lambda & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -\lambda \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

- As last computation, let us consider the root $-3\alpha_1 - 2\alpha_2$. By (6.2.2) and the condition for $u_{-3\alpha_1-2\alpha_2}$ to be a group homomorphism, there exist some constants η_1 and $\eta_2 \in k$ such that for any $\lambda \in \mathbf{G}_a$, $u_{-3\alpha_1-2\alpha_2}(\lambda)$ acts on V as

$$u_{-3\alpha_1-2\alpha_2}(\lambda) \cdot x = (x_0, x_1 + \eta_1\lambda x_2, x_2, x_3, x_4 + \eta_2\lambda x_5, x_5, x_6).$$

The isometry condition implies

$$\begin{aligned} q(x) = q(u_{-3\alpha_1-2\alpha_2}(\lambda) \cdot x) &= -x_3^2 - x_2x_4 - \eta_2\lambda x_2x_5 - x_1x_5 - \eta_1\lambda x_2x_5 - x_0x_6 \\ &= q(x) - (\eta_2 + \eta_1)\lambda x_2x_5, \end{aligned}$$

hence $\eta_2 = -\eta_1$ and we can conclude that the root subgroups have the following form :
(6.2.3)

$$u_{-3\alpha_1-2\alpha_2} \colon \lambda \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & \lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -\lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad u_{3\alpha_1+2\alpha_2} \colon \lambda \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\lambda & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Remark 6.2.1. Let us recall that in characteristic 2 the group G acts on $W = V/ke$, giving an embedding $G \subset \mathrm{Sp}_6$: we list below what the root subspaces we need become in that case.

$$u_{\alpha_1}(\lambda) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \lambda^2 \\ 0 & 1 & 0 & \lambda & 0 & 0 \\ 0 & 0 & 1 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad u_{-\alpha_1}(\lambda) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 1 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 1 & 0 \\ \lambda^2 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$u_{\alpha_2}(\lambda) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \lambda & 0 & 0 & 0 & 1 & 0 \\ 0 & \lambda & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad u_{-\alpha_2}(\lambda) = \begin{pmatrix} 1 & 0 & 0 & 0 & \lambda & 0 \\ 0 & 1 & 0 & 0 & 0 & \lambda \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$u_{2\alpha_1+\alpha_2}(\lambda) = \begin{pmatrix} 1 & \lambda & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \lambda^2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \lambda \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad u_{-2\alpha_1-\alpha_2}(\lambda) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \lambda & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \lambda^2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \lambda & 1 \end{pmatrix}$$

$$u_{\alpha_1+\alpha_2}(\lambda) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \lambda & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & \lambda^2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \lambda & 0 & 1 \end{pmatrix}, \qquad u_{-\alpha_1-\alpha_2}(\lambda) = \begin{pmatrix} 1 & 0 & \lambda & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \lambda^2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \lambda \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$u_{3\alpha_1+2\alpha_2}(\lambda) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & \lambda & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \lambda & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad u_{-3\alpha_1-2\alpha_2}(\lambda) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & \lambda & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \lambda & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

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COLLEGE DOCTORAL BRETAGNE
MATHS, TELECOMS
INFORMATIQUE, SIGNAL
SYSTEMES, ELECTRONIQUE

Université de Rennes

Titre : Sous-schémas en groupes paraboliques et variétés homogènes en petites caractéristiques

Mot clés : caractéristique positive, groupes algébriques semi-simples, sous-groupes paraboliques, variétés projectives homogènes

Résumé : Cette thèse achève la classification des sous-schémas en groupes paraboliques des groupes algébriques semi-simples sur un corps algébriquement clos, en particulier de caractéristique deux et trois. Dans un premier temps, nous présentons la classification en supposant que la partie réduite de ces sous-groupes soit maximale, avant de passer au cas général. Nous parvenons à une description quasiment uniforme : à l'exception d'un groupe de type G_2 en caractéristique deux, chaque sous-schémas en groupes parabolique est obtenu en multipliant des paraboliques réduits par des noyaux d'isogénies purement inséparables, puis en prenant l'intersection. En conclusion, nous discutons quelques implications géométriques de cette classification.

Title: Parabolic subgroup schemes and homogeneous varieties in small characteristics

Keywords: positive characteristic, semisimple algebraic groups, parabolic subgroups, projective homogeneous varieties

Abstract: This thesis brings to an end the classification of parabolic subgroup schemes of semisimple groups over an algebraically closed field, focusing on characteristic two and three. First, we present the classification under the assumption that the reduced part of these subgroups is maximal; then we proceed to the general case. We arrive at an almost uniform description: with the exception of a group of type G_2 in characteristic two, any parabolic subgroup scheme is obtained by multiplying reduced parabolic subgroups by kernels of purely inseparable isogenies, then taking the intersection. In conclusion, we discuss some geometric implications of this classification.